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ABSTRACT

This is part two of a two-part SMSG text for grade seven students whose mathematical talents are underdeveloped. The reading level of this text has been adjusted downward, chapters shortened, and additional concrete examples included. Nevertheless, the authors warn that the text may not be appropriate for the very slow non-college-bound student. Chapter topics include: rational numbers and fractions, non-metric geometry, rational numbers and the number line, decimals, and ratio and percent. (MN)

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INTRODUCTION TO SECONDARY SCHOOL MATHEMATICS
VOLUME 1
PART 1

SE 024 958

SCHOOL MATHEMATICS STUDY GROUP

School Mathematics Study Group

Introduction to Secondary School Mathematics, Volume I

Student's Text, Part II

Unit 38

REVISED EDITION

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Chapter 6

RATIONAL NUMBERS AND FRACTIONS

6-1. An Invitation to Pretend.

In very early times, the only numbers used were the counting numbers. As the need arose, rules were developed for working with the counting numbers. Much later, the general principles, upon which the operations with fractions are based, were discovered.

As you study the material in this chapter, it will help if you will pretend that it is all new to you. Pretend that you have never heard of any numbers except the whole numbers. Try to think as if you were one of the first to realize that the whole numbers are not enough.

When new numbers are introduced, symbols must be invented for them. Questions about how to define multiplication and addition of the new numbers must be answered. These are the ideas which you will study in this chapter.

Consider the following examples:

Example 1: Eggs cost 60 cents a dozen. How much does one egg cost?

You know that you can find the cost of one egg by dividing the number of cents by the number of eggs. The work looks like this.

$$60 \div 12 = 5$$

Example 2: Eggs cost 53 cents a dozen. How much does one egg cost?

Here again, you divide the number of cents by the number of eggs.

$$53 \div 12 = ?$$

In the second example, you see that 53 cannot be exactly divided by 12.

If you had never heard of fractions you would say, "You can't divide 53 by 12." This would be correct. You would mean that, in the set of counting numbers, there is no number which is equal to $53 \div 12$. You know that the set of counting numbers is not closed under division. This means that when you divide one counting number by another you do not always get an answer which is a counting number. Thus, in order to solve problems like Example 2 it is necessary to use a new kind of number.

Exercises 6-1

1. Just as you have done for multiplication and addition, now make a division table. In each box place the result of dividing the number on the left by the number on the top if this quotient is a whole number. If the quotient is not a whole number write "no." Some of the blanks have been filled to help you understand what is meant.

÷	1	2	3	4	5	6	7	8
1	1	no	no					
2	2	1	no					
3	3	no						
4								
5								
6		3						
7								
8					no			

2. Jim spent 90¢ for chocolate bars. They cost 5¢ each. How many did he buy? He shared them equally with his friends, Jack and Harry. How many bars did each boy receive? Could he have shared the bars equally if Mark had been present also?

3. a. The gym teacher divided sixteen boys into two equal teams to play a game. How many boys were on each team? How did you solve this problem?
   b. The gym teacher divided seventeen boys into two equal teams to play a game. How many boys were on each team?

6-2. The Invention of the Rational Numbers.

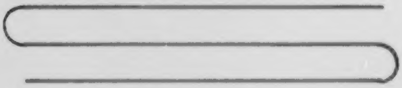
In the last section, you saw that the set of counting numbers is not closed under division; that is, the quotient of two counting numbers is not always a counting number. Let us consider the following problem.

> Example: Mrs. Green has a ribbon 100 inches long. She wants to divide it equally among her 3 daughters. How long a piece of ribbon should each daughter get?

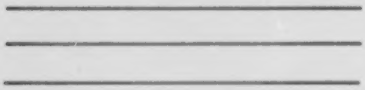
You may obtain the answer as follows:

$$100 \div 3 = ?$$

Think how hard this problem must have seemed in the days when fractions had not been invented. Mrs. Green tried to solve the problem by folding the ribbon into 3 pieces of equal length like this.



Then she cut the ribbon at the folds to have 3 pieces of equal length.



Each piece had a definite length, but this length could not be given by any counting number. She would have liked to have a number to describe this length. What could she have done? She could have overcome this difficulty by inventing a new number.

How could this number have been named? It seemed reasonable that the new name should use the numbers 100 and 3 in some way. She decided to give this number the name " $\frac{100}{3}$ ". In her mind, she thought, " $\frac{100}{3}$ is the number obtained when 100 is divided by 3."

You are familiar with symbols like $\frac{100}{3}$ and you know that $\frac{100}{3}$ means the quotient when 100 is divided by 3. Another possible symbol might be (100, 3). You may have read that $\frac{100}{3}$

is an "indicated" quotient. It actually __is__ a quotient. So are the numbers $12\overline{)16}$, $19 \div 2$, $\frac{4}{7}$.

The numbers which we get when we divide a whole number by a counting number have a name. They are called __rational numbers__. The word "rational" is derived from the word "ratio" which is another word for quotient. Note that we must speak of dividing by a counting number rather than by a whole number since we cannot divide by zero. Some examples of rational numbers are $\frac{3}{4}$, $\frac{17}{9}$, $\frac{0}{8}$, $\frac{15}{3}$.

__Exercises 6-2__

1. Explain the meaning of $\frac{7}{9}$, $\frac{15}{2}$, $\frac{43}{3}$, $\frac{17}{17}$, $\frac{29}{14}$.

   For example: $\frac{7}{9}$ means 7 divided by 9.

2. $\frac{26}{2}$ names a rational number. This rational number is also a __________ number. Some rational numbers are __________ numbers.

3. Express the quotient of 13 divided by 9 in three ways.

4. Three men who took a trip of 750 miles shared the driving equally. How far did each man drive?

5. Three men who took a trip of 700 miles shared the driving equally. How far did each man drive?

6. A bus carrying 50 passengers made a trip of 500 miles. How far did each passenger travel?

7. Perform the following division and check your work by multiplication.

   Arrange your work like this:

$$\begin{array}{r} 65 \\ 42\overline{)2730} \\ \underline{252} \\ 210 \\ \underline{210} \\ 0 \end{array} \qquad \text{Check:} \quad \begin{array}{r} 65 \\ \underline{42} \\ 130 \\ \underline{260} \\ 2730 \end{array}$$

$$2730 \div 42 = 65 \qquad 42 \cdot 65 = 2730$$

a. $2752 \div 43$
b. $2915 \div 53$
c. $2916 \div 54$
d. $3293 \div 37$
e. $2093 \div 23$
f. $2541 \div 33$
g. $1729 \div 19$
h. $13431 \div 121$
i. $1001 \div 13$

8. During a recent year, 8133 ocean-going vessels passed through the Panama Canal.
   a. About how many vessels passed through each week, on the average?
   b. One of the vessels had been at sea for 5400 hours. How many days (24 hours) was this?

6-3. Fractions and Rational Numbers.

In the last section we saw how the need for rational numbers arose. These numbers give us an answer to any division problem where we divide a whole number by a counting number. For example, $14 \div 7 = \frac{14}{7}$. But we know that $14 \div 7 = 2$. Perhaps, you will say that we can get two different answers to the problem of dividing 14 by 7. This is wrong. The fact is that $\frac{14}{7}$ and 2 are the same numbers. You may say that "$\frac{14}{7}$" and "2" look different. How can they be the same? The things that look different are the symbols we write down. These symbols are the names for the numbers. The symbols are not the numbers. This point was stressed in Chapter 2. Symbols for numbers are called numerals.

$$\text{"6"}, \quad \text{"VI"}, \quad \text{"}11_{\text{five}}\text{"}, \quad \text{"}7 - 1\text{"}$$

are different names for the number 6. The numerals

$$\text{"}\frac{12}{2}\text{"}, \quad \text{"}\frac{18}{3}\text{"}, \quad \text{"}\frac{6}{1}\text{"}, \quad \text{"}24 \div 4\text{"}, \quad \text{"}7\overline{)42}\text{"}$$

are also names for the number 6. It is important to know when we are talking about a number and when we are talking about a name for a number.

In this chapter you will frequently find the words "fraction" and "rational number". Let us consider the difference in meaning between "fraction" and "rational number".

The numerals "$\frac{15}{7}$", "$\frac{6}{2}$", "$\frac{12}{3}$" are called fractions.

The numeral "$\frac{a}{b}$" where a is a whole number and b is a counting number is called a fraction. Fractions are symbols for numbers. When we write

$$\frac{14}{7} = \frac{2}{1}$$

we are not saying that the fractions "$\frac{14}{7}$" and "$\frac{2}{1}$" are the same symbols. We are saying that the numbers which these fractions stand for are the same. We are fussy about this in order to avoid misunderstanding later.

Every rational number has a name which is a fraction. Some rational numbers have other names which are not fractions. The number 2 is a rational number. Its numeral "2" is not a fraction because it is not of the form $\frac{a}{b}$. Other names for 2 are "$\frac{14}{7}$", "$\frac{6}{3}$", "$\frac{2}{1}$". All of these names are fractions. It is correct to say that rational numbers are numbers which have fractions as names. The number 2 has a fraction as a name as we have just seen. Therefore, 2 is rational. So are all the other whole numbers.

As you study more mathematics, you will learn how the number system is extended to include numbers which do not have fractions as names. Some of these numbers will be needed when you study circles and right triangles.

Exercises 6-3a

1. a. Is $\frac{0}{5}$ a fraction?
   b. What other name which is not a fraction does the number $\frac{0}{5}$ have?
   c. Is zero a rational number?
2. a. Is $4\overline{)3}$ a fraction?
   b. Does $4\overline{)3}$ name a rational number?
3. Does the statement, "Every rational number can be expressed as a fraction" mean that rational numbers have no other names?
4. Tell which of the following represent rational numbers:
   a. $\frac{1}{6}$
   b. $14\overline{)63}$
   c. $\frac{5}{1}$
   d. $8 + 7$
   e. 2.15
   f. $\frac{9}{11}$
5. Which of the numerals in Problem 4 are fractions?

6. a. Select from the following set of numerals those which are fractions: 8, $\frac{56}{7}$, $16 \div 2$, $3\overline{)24}$, 8.00 .

   b. Select from the given set the numerals which name rational numbers.

7. Express the answers to the following questions by using fractions and also without using fractions.

   a. Three men who took a trip of 750 miles divided the driving equally. How far did each man drive?

   b. A woman divided 90 inches of ribbon equally among 3 daughters. How much did each daughter get?

   c. A dozen eggs cost 60 cents. How much did each egg cost?

8. Express the answers to the following, using fractions. Can you express the answers without using fractions?

   a. Three men who took a trip of 700 miles divided the driving equally. How far did each man drive?

   b. A woman divided 100 inches of ribbon equally among 3 daughters. How much did each daughter get?

   c. A dozen eggs cost 43 cents. How much did each egg cost?

9. Consider the following set of symbols:

   $\{37, 1, 0, \frac{2}{3}, 2\frac{1}{2}, \frac{0}{3}, \frac{111}{3}, \frac{10}{0}, \frac{7}{7}, \frac{25}{10}, \frac{5}{2}, \frac{0}{0}\}$

   a. Which of the above are numerals for counting numbers?

   b. Which of the above are numerals for whole numbers?

   c. Which of the above are numerals for rational numbers?

   d. Which of the above are fractions?

   e. Which of the above are meaningless symbols?

   f. Which of the above are different names for the same number?

Exercises 6-3b

(Class Discussion)

1. a. What number must you multiply 3 by to get 90 ?

   b. What number must you multiply 12 by to get 60 ?

   c. What number must you multiply 3 by to get 750 ?

   d. What number must you multiply 18 by to get 54 ?

   e. What number must you multiply 11 by to get 5280 ?

   f. What operation did you use to find the answers?

2. Can you find a number to put in the blank so that the statement will be true?

   a. $3 \cdot ? = 90$
      $90 \div 3 = ?$

   b. $12 \cdot ? = 60$
      $60 \div 12 = ?$

   c. $3 \cdot ? = 750$
      $750 \div 3 = ?$

   d. $18 \cdot ? = 54$
      $54 \div 18 = ?$

   e. $11 \cdot ? = 5280$
      $5280 \div 11 = ?$

   f. $19 \cdot ? = 1729$
      $1729 \div 19 = ?$

3. In each of the following problems the symbol n stands for a number. In each case tell what number n must stand for.

   a. $3 \cdot n = 15$
   b. $4 \cdot n = 20$
   c. $7 \cdot n = 56$
   d. $18 \cdot n = 54$
   e. $19 \cdot n = 1729$
   f. $13 \cdot n = 1001$
   g. $17 \cdot n = 17$
   h. $14 \cdot n = 0$

4. In Problem 3, what operation did you use to find each answer?

6-4. The Meaning of Division.

In Chapter 3 you learned that division by a number is the inverse of multiplication by the same number:

1. $\underline{\underline{12}} \div \underline{4} = 3$ Divide $\underline{\underline{12}}$ by $\underline{4}$ and get 3.
2. $\underline{4} \cdot 3 = \underline{\underline{12}}$ Multiply 3 by $\underline{4}$ and get $\underline{\underline{12}}$.

The two statements say the same thing in different ways; so do the following ways of expressing the idea:

3. $12 \div 4 = ?$ What is the result of dividing 12 by 4?
4. $4 \cdot ? = 12$ What number multiplied by 4 gives 12 ?

We have two kinds of problems using products.

One kind is like this: $4 \cdot 5 = ?$
We choose to write it like this: $4 \cdot 5 = n$ } Multiplication

Both statements ask: What is the product?

This is called a multiplication problem.

$$\left.\begin{array}{ll}\text{The other kind is like this:} & 4\cdot ? = 20 \\ \text{We choose to write it like this:} & 4\cdot y = 20\end{array}\right\}\quad\text{Division}$$

Both statements ask: What is the number you multiply by 4 to get 20 ?

This is called a division problem.

To get the answer we must divide even though the division sign is not used. We ask, "What should be the number y so that $4 \cdot y$ will be 20 ?" This number is called the quotient of 20 by 4 or $20 \div 4$ or $\frac{20}{4}$.

In this way we can write division problems using only the multiplication sign.

The statements $4 \cdot y = 20$

and $y = \frac{20}{4}$

tell us that $\frac{20}{4}$ is the number y for which $4 \cdot y = 20$.

In the same way, $\frac{4}{3}$ is the number t for which $3 \cdot t = 4$.

Exercises 6-4

1. Copy and complete the following sentences:
   - a. What number do you multiply by 9 to get 8 ?
     $9 \cdot ? = 8$
   - b. What number do you multiply by 4 to get 7 ?
     $4 \cdot ? = 7$
   - c. $\frac{6}{3}$ is the number x for which $3 \cdot x = ?$
   - d. $\frac{5}{2}$ is the number x for which $? \cdot x = ?$
   - e. $\frac{9}{3}$ is the number x for which $? \cdot x = ?$
   - f. $\frac{18}{6}$ is the number x for which $? \cdot x = ?$
   - g. $\frac{10}{4}$ is the number x for which $? \cdot x = ?$
   - h. $\frac{7}{1}$ is the number x for which $? \cdot x = ?$
2. In which parts of Problem 1 is the number x a whole number? Find the number for x whenever it is a whole number.

3. In each of the following sentences, find the number r which makes the statement true. (Keep the fraction form.)

a. $2 \cdot r = 3$
b. $5 \cdot r = 40$
c. $4 \cdot r = 15$
d. $7 \cdot r = 56$
e. $3 \cdot 4 = 21$
f. $5 \cdot r = 11$
g. $10 \cdot r = 60$
h. $8 \cdot r = 1$

4. For each of the following write a sentence which describes the problem in mathematical language. Use a letter, such as n, for the unknown number and tell, in words, what it stands for in each case.

Example: Sam's father is sawing a log 12 feet long into 6 equal lengths. How long will each piece be?

Answer: If x is the number of feet in each piece, then $6 \cdot x = 12$.

a. If 12 cookies are divided equally among 3 boys, how many cookies should each boy receive?
b. Mr. Carter's car used 10 gallons of gasoline for a 160 mile trip. How many miles did he drive for each gallon of gasoline used?
c. If it takes 20 bags of cement to make a 30 foot walk, how much cement is needed for each foot of the walk?

5. Without dividing or factoring, decide which of the following statements are true. As an example, to show that $\frac{168}{21} = 8$, multiply 8 by 21 to find out if the product is 168.

a. $\frac{169}{13} = 13$
b. $\frac{262}{17} = 16$
c. $\frac{744}{124} = 6$
d. $\frac{143}{11} = 13$
e. $\frac{15251}{151} = 101$

6-5. Rational Numbers in General.

In the last section we said

$\frac{20}{4}$ is the number n for which $4 \cdot n = 20$,

$\frac{4}{3}$ is the number n for which $3 \cdot n = 4$.

From these examples, can we say exactly what $\frac{17}{9}$, $\frac{1}{5}$, $\frac{2134}{193491}$, and $\frac{0}{3}$ mean? We can, but not in a single statement that includes all of the examples at one time. The task is easier if we use letters to stand for numbers. Here is a way that we could say this with words. (Don't try to remember it. It is only put in as a horrible example.)

> If we write the numerals for two counting numbers, one above the other, with a horizontal line between them, then the resulting symbol is a numeral for the number which, when multiplied by the number whose numeral we have written in the bottom, gives the number whose numeral we have written on top.

Here is the way that we say this if we use letters to stand for numbers.

Definition: If a and b are whole numbers $(b \neq 0)$ then $\frac{a}{b}$ is the number n for which $b \cdot n = a$.

Clearly, the second statement is easier and more convenient.

In the definition, if you let $a = 20$ and $b = 4$, you get $\frac{a}{b} = \frac{20}{4}$.

Then $\frac{20}{4}$ is the number n for which $4 \cdot n = 20$, and $4 \cdot \frac{20}{4} = 20$.

Examples: $\frac{5}{9}$ is the number for which $9 \cdot \frac{5}{9} = 5$.
$\frac{13}{2}$ is the number for which $2 \cdot \frac{13}{2} = 13$.
$\frac{7}{16}$ is the number for which $? \cdot \frac{7}{16} = 7$.
$\frac{9}{16}$ is the number for which $16 \cdot \frac{9}{16} = ?$

When you write "$\frac{a}{b}$ is the number n for which $b \cdot n = a$" you are using "n" to stand for "$\frac{a}{b}$". Then $b \cdot n = a$ becomes

$$b \cdot \frac{a}{b} = a$$

when you replace "n" by "$\frac{a}{b}$".

For example: $2 \cdot \frac{6}{2} = 6$ and $3 \cdot \frac{4}{3} = 4$.

Since you are just beginning to investigate fractions and rational numbers, all that you know about them is that:

1. Every rational number has a name which is a fraction.
2. The rational number $\frac{a}{b} (b \neq 0)$ is the number for which

$$b \cdot \frac{a}{b} = a.$$

In this new look at fractions you are asked to pretend that you do not know the meaning of addition or multiplication of rational numbers or the manipulation of the symbols which produces fractions for sums and products of rational numbers.

Exercises 6-5

Practice using the pattern $b \cdot \frac{a}{b} = a$ in the following problems.

Example: $2 \cdot ? = 6$ is $2 \cdot \frac{6}{2} = 6$.

1. Copy and complete.

    a. $3 \cdot ? = 6$
    b. $5 \cdot ? = 50$
    c. $7 \cdot ? = 63$
    d. $12 \cdot ? = 132$
    e. $19 \cdot ? = 1729$
    f. $35 \cdot ? = 1960$

2. In each case find the value of x which makes the statement correct.

    a. $3 \cdot x = 6$
    b. $5 \cdot x = 50$
    c. $7 \cdot x = 63$
    d. $12 \cdot x = 132$
    e. $19 \cdot x = 1729$
    f. $35 \cdot x = 1960$

3. In each case find the value of x. Express the answer as a fraction.

    a. $3 \cdot x = 7$
    b. $5 \cdot x = 4$
    c. $7 \cdot x = 5$
    d. $12 \cdot x = 6$
    e. $19 \cdot x = 29$
    f. $35 \cdot x = 345$

4. Copy and complete.

    a. $5 \cdot \frac{6}{5} = ?$
    b. $12 \cdot \frac{7}{12} = ?$
    c. $9 \cdot \frac{6}{9} = ?$
    d. $3 \cdot \frac{5}{3} = ?$
    e. $5 \cdot \frac{10}{5} = ?$
    f. $14 \cdot \frac{13}{14} = ?$

5. Copy and complete.

a. $? \cdot \frac{7}{5} = 7$

b. $? \cdot \frac{5}{8} = 5$

c. $? \cdot \frac{3}{9} = 3$

d. $? \cdot \frac{14}{5} = 14$

e. $? \cdot \frac{11}{9} = 11$

f. $? \cdot \frac{9}{11} = 9$

6. Copy and complete.

a. $6 \cdot \frac{?}{6} = 5$

b. $12 \cdot \frac{?}{12} = 6$

c. $9 \cdot \frac{7}{?} = 7$

d. $4 \cdot \frac{9}{?} = 9$

e. $14 \cdot \frac{7}{?} = 7$

f. $11 \cdot \frac{?}{?} = 9$

g. $16 \cdot \frac{?}{?} = 1$

h. $4 \cdot \frac{?}{?} = 1$

i. $1739 \cdot \frac{?}{?} = 2403$

j. $609253 \cdot \frac{?}{?} = 17963$

7. Copy and complete. Use the pattern $b \cdot \frac{a}{b} = a$.

a. $? \cdot \frac{6}{5} = ?$

b. $? \cdot \frac{9}{10} = ?$

c. $? \cdot \frac{3}{6} = ?$

d. $? \cdot \frac{24}{16} = ?$

e. $? \cdot \frac{1}{12} = ?$

f. $? \cdot \frac{0}{7} = ?$

g. $? \cdot \frac{5}{1} = ?$

h. $? \cdot \frac{93147}{62973} = ?$

8. **BRAINBUSTER:** Each letter is a different digit and E is greater than 2.

$$\frac{\text{MATH}}{\text{EXAM}} = \text{E}$$

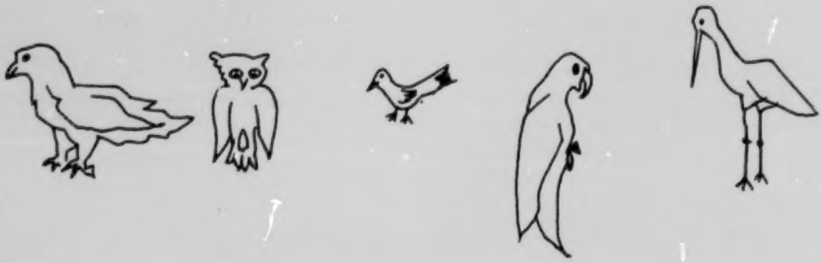
What numerals containing four digits do MATH and EXAM represent?

6-6. Properties of Operations with Rational Numbers.

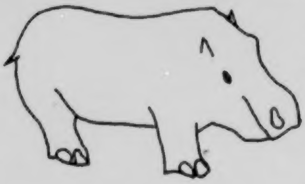
We have now extended the number system to include rational numbers. We found it necessary to do this in order to be able to divide whole numbers. Now we shall ask questions about multiplication, division, addition, and subtraction of the rational numbers.

The first problem will be to extend the meaning of the operation of multiplication. Using only what we know about the multiplication of whole numbers we shall show how the operation of multiplication is extended to the rational numbers. Let us think about the idea of extending the meaning of a word.

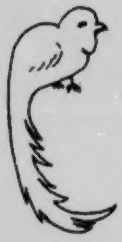
Suppose you had specimens of each of the following: eagle, owl, sparrow, parrot, stork. Suppose you decided to call all these creatures birds.



Now suppose that someone came to you with some other creature and asked you whether it was a bird. Suppose he pointed out a hippopotamus.



Would you call this creature a bird? Certainly not. Suppose someone came with the creature pictured below.



Would you call this creature a bird? You would, wouldn't you?

How did you make these decisions? From your limited knowledge of birds how did you decide to extend the idea of bird to include the quetsal but not to include the hippopotamus? You probably made this decision very quickly because you have been doing this sort of thing all your life. But there was a thought process involved that went like this:

> The five given examples of living creatures that are called birds have certain properties in common. Among these properties are: wings, feathers, beaks, two legs, claws.
>
> You decided to extend the use of the name "bird" to include any creatures that have these properties, but not to include creatures that do not have these properties.

Now what does all this have to do with mathematics? Simply this. We are going to extend the operation of multiplication to operate on rational numbers. And we want to do this in such a way that we preserve the properties of multiplication of whole numbers. We have already defined the operations of multiplication for certain basic products like

$$5 \cdot \frac{3}{5} = 3, \qquad 2 \cdot \frac{7}{2} = ?, \qquad 45 \cdot \frac{11}{45} = ?$$

which follow the pattern $b \cdot \frac{a}{b} = a$.

But how shall we define products which are not of this type, as for example:

$$5 \cdot \frac{3}{4}, \qquad \frac{6}{7} \cdot \frac{3}{5}, \qquad \frac{4}{3} \cdot \frac{3}{8}$$

We might think that we can define these products to be anything we like. But just as in the case of the birds, we wish to extend the idea of multiplication to the rational numbers in a way that preserves the properties of multiplication.

You learned about the basic properties of these operations with whole numbers in Chapter 3. They are listed again below.

Addition	Multiplication
Closure Property $a + b$ is a number	Closure Property $a \cdot b$ is a number
Commutative Property $a + b = b + a$	Commutative Property $a \cdot b = b \cdot a$
Associative Property $a + (b + c) = (a + b) + c$	Associative Property $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
Distributive Property $a(b + c) = (a \cdot b) + (a \cdot c)$	
Identity Property of 0 $a + 0 = 0 + a = a$	Identity Property of 1 $a \cdot 1 = 1 \cdot a = a$
	Multiplication Property of 0 $a \cdot 0 = 0 \cdot a = 0$

These properties held true when the numbers we had were the whole numbers. Now we wish to require that these properties hold true when the numbers we have are the rational numbers. What does this requirement mean? Some specific examples of what it means are that

$\frac{2}{3} \cdot \frac{4}{5}$ must be a rational number (closure for multiplication)

$\frac{2}{3} + \frac{4}{5}$ must be a rational number (closure for addition)

$\frac{2}{3} \cdot \frac{4}{5}$ must be the same number as $\frac{4}{5} \cdot \frac{2}{3}$ Which property is illustrated?

$\frac{2}{3} \cdot 1$ must be the same number as $\frac{2}{3}$ Which property is illustrated?

Exercises 6-6

1. Illustrate the following properties from the list on page 200 as properties of rational numbers like $\frac{2}{3}, \frac{3}{4}, \frac{7}{5}$.
    a. Associative Property of Addition
    b. Distributive Property
    c. Identity Property of 1
    d. Commutative Property of Multiplication

2. Give an example of each of the following:
    a. Counting number.
    b. Whole number.
    c. A whole number which is not a counting number.
    d. A rational number which is not a whole number.
    e. A numeral which is not a fraction but names a rational number.

3. Which of the following represent rational numbers?
    a. $\frac{3}{7}$
    b. $\frac{5}{3}$
    c. $\frac{1}{10}$
    d. 4
    e. 1
    f. $\frac{16}{4}$
    g. 0.2
    h. 0.13

4. Copy and complete, or find the number that x represents.
    a. $15 \cdot x = 14$
    b. $15 \cdot \frac{?}{?} = 14$
    c. $\_\_\_ \cdot \frac{14}{15} = 14$
    d. $15 \cdot \frac{14}{15} = \_\_\_$
    e. $\_\_\_ \cdot \frac{14}{15} = \_\_\_$

5. a. $3 \cdot x = 4$

   b. $3 \cdot \frac{?}{?} = 5$

   c. $4 \cdot \frac{?}{?} = 5$

   d. $7 \cdot \frac{6}{7} = \_\_\_$

   e. $\_\_\_ \cdot \frac{5}{8} = 5$

   f. $15 \cdot x = 13$

   g. $\_\_\_ \cdot \frac{7}{4} = \_\_\_$

   h. $6 \cdot x = 5$

   i. $9 \cdot \frac{?}{?} = 7$

   j. $\_\_\_ \cdot \frac{18}{11} = \_\_\_$

   k. $\_\_\_ \cdot \frac{1}{6} = \_\_\_$

   l. $5 \cdot \frac{?}{?} = 1$

   m. $5 \cdot x = 1$

   n. $17 \cdot \frac{?}{?} = 0$

6. **Matching.** Each example on the left illustrates one of the properties listed on the right. Write the number of the property beside the example which describes it.

   a. $\frac{2}{3} \cdot \frac{5}{6} = \frac{5}{6} \cdot \frac{2}{3}$

   b. $\frac{5}{2}(\frac{3}{2} \cdot \frac{2}{5}) = (\frac{5}{2} \cdot \frac{3}{2}) \cdot \frac{2}{5}$

   c. $5(2 + 3) = 5 \cdot 2 + 5 \cdot 3$

   d. $\frac{1}{2}(4 + 6) = \frac{1}{2}(4) + \frac{1}{2}(6)$

   e. $1 \cdot \frac{5}{6} = \frac{5}{6}$

   f. $\frac{2}{3} \cdot \frac{5}{5} = \frac{2}{3}$

   g. $\frac{7}{8} + 0 = \frac{7}{8}$

   h. $\frac{3}{2} + \frac{4}{5} = \frac{4}{5} + \frac{3}{2}$

   i. $\frac{5}{6} \cdot 0 = 0$

   j. $\frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) = (\frac{1}{2} + \frac{1}{3}) + \frac{1}{4}$

   (1) Commutative property of addition

   (2) Commutative property of multiplication

   (3) Associative property of addition

   (4) Associative property of multiplication

   (5) Distributive property of multiplication over addition

   (6) Identity property of 0

   (7) Identity property of 1

   (8) Multiplication property of 0

6-7. Multiplication of Rational Numbers.

A fraction is a numeral which is written in a specific way. The symbol "$\frac{2}{3}$" is a fraction. A symbol for the product of two rational numbers, $\frac{2}{3}$ and $\frac{7}{5}$ is "$\frac{2}{3} \cdot \frac{7}{5}$." This symbol "$\frac{2}{3} \cdot \frac{7}{5}$" is not a fraction.

If you knew nothing at all about how to find a fraction for the product $\frac{2}{3} \cdot \frac{7}{5}$, you might guess that a possible numeral for this product could be "$\frac{14}{15}$." Someone else might suggest "$\frac{27}{35}$" or even "$\frac{10}{21}$." Why is "$\frac{14}{15}$" the one which is accepted and not the others? You may even ask what right we have to deal separately with the parts of the symbols in $\frac{2}{3} \cdot \frac{7}{5}$ since each fraction stands for a distinct rational number.

Our problem is to define what we mean by multiplication of rational numbers and at the same time to find out what to do with the numerals. We are free to make any definitions we want to make. It is our intention to make a definition that will be useful and will fit with what we already know about whole numbers.

Therefore, we shall define a product like $(\frac{2}{3} \cdot \frac{7}{5})$ so that the commutative and associative properties are kept. We shall begin with what we know about the whole numbers and rational numbers. We shall extend the properties of multiplication of whole numbers to the multiplication of rational numbers and observe what happens.

Exercises 6-7a

(Class Discussion)

This set of exercises is different from other sets you have had. It presents a big idea in small steps. The correct answers are given at the right and you are asked to cover all of them until you have made your responses to the first one. Then look at the correct answer to the first question. After each response you make, look at the correct answer.

Since these are class discussion questions, you will be able to ask questions to clarify points you do not understand. Be sure that each step is clear before you go on to the next one.

	Cover
1. $\frac{2}{3}$ and $\frac{7}{5}$ are both ________ numbers. We wish to find a single fraction for the product $\frac{2}{3} \cdot \frac{7}{5}$ if there is such a numeral.	rational
2. To find out what we can about the product $\frac{2}{3} \cdot \frac{7}{5}$ we use what we know about $\frac{2}{3}$ and $\frac{7}{5}$ and also the properties of multiplication which we wish to keep. From $b \cdot \frac{a}{b} = a$ we know that $\underline{3 \cdot \frac{2}{3} = \qquad}$.	$3 \cdot \frac{2}{3} = 2$
3. We also know that $\underline{\quad \cdot \frac{7}{5} = \qquad}$.	$5 \cdot \frac{7}{5} = 7$
4. When we say $3 \cdot \frac{2}{3} = 2$ we mean that "$3 \cdot \frac{2}{3}$" and "______" are symbols for the same number. The symbols "$3 \cdot \frac{2}{3}$" and "2" are different pencil scratches, but they name the same number.	2
5. Also, the same number is named by the pencil scratches "$5 \cdot \frac{7}{5}$" and "____".	7

6. Here are two products that look different: $(3 \cdot \frac{2}{3}) \cdot (5 \cdot \frac{7}{5})$ and $2 \cdot 7$ We know that we multiply numbers, not pencil scratches. Therefore, the symbols that look different are simply two ways of writing the s_____ product and we may write $(3 \cdot \frac{2}{3}) \cdot (5 \cdot \frac{7}{5}) = 2 \cdot \underline{\qquad}$.	same product $2 \cdot 7$
7. Another symbol for "$2 \cdot 7$" is "14". Then we may write $(3 \cdot \frac{2}{3}) \cdot (5 \cdot \frac{7}{5}) = \underline{\qquad}$.	14
8. We should like to group the numbers in $(3 \cdot \frac{2}{3}) \cdot (5 \cdot \frac{7}{5})$ in another way. We are permitted to group these factors in a different way by the commutative and __________ properties of multiplication. $(3 \cdot \frac{2}{3}) \cdot (5 \cdot \frac{7}{5}) = \underline{(\quad \cdot \quad)} \cdot (\frac{2}{3} \cdot \frac{7}{5})$	associative $(3 \cdot 5)$
9. Then $(3 \cdot 5) \cdot \underline{(\quad \cdot \quad)} = 14$ and since $(3 \cdot 5)$ is 15 we write $15 \cdot (\frac{2}{3} \cdot \frac{7}{5}) = 14$	$(\frac{2}{3} \cdot \frac{7}{5})$
10. Compare the products $15 \cdot (\frac{2}{3} \cdot \frac{7}{5}) = 14$ and $b \cdot n = a$ We can match these exactly if we think b is 15 n is $\underline{(\qquad)}$ a is ____	$(\frac{2}{3} \cdot \frac{7}{5})$ 14

11. We know that if $b \cdot n = a$ then $\underline{n = \frac{\quad}{b}}$	$n = \frac{a}{b}$
12. The matching we did in Step 10 tells us that $\underline{\frac{a}{b} = \frac{\quad}{15}}$.	$\frac{a}{b} = \frac{14}{15}$
13. Then for $n = \frac{a}{b}$ we may write $\underline{(\frac{2}{3} \cdot \frac{7}{5}) = \frac{\quad}{15}}$ We have found that $\frac{2}{3} \cdot \frac{7}{5} = \frac{14}{15}$ or $\frac{2}{3} \cdot \frac{7}{5} = \frac{2 \cdot 7}{3 \cdot 5}$	$(\frac{2}{3} \cdot \frac{7}{5}) = \frac{14}{15}$

This result may not be a great surprise to you. The point is that you have been using a procedure on faith. Now you know that this procedure is based on mathematical reasons. It is not merely an accident, but a logical result.

In short form the steps you just went through can be written this way:

$$3 \cdot \frac{2}{3} = 2 \quad \text{and} \quad 5 \cdot \frac{7}{5} = 7$$

$$(3 \cdot \frac{2}{3}) \cdot (5 \cdot \frac{7}{5}) = 2 \cdot 7 \quad \text{and} \quad 2 \cdot 7 = 14$$

$$(3 \cdot 5) \cdot (\frac{2}{3} \cdot \frac{7}{5}) = 14$$

$$15 \cdot (\frac{2}{3} \cdot \frac{7}{5}) = 14$$

$$\frac{2}{3} \cdot \frac{7}{5} = \frac{14}{15} \quad \text{or} \quad \frac{2}{3} \cdot \frac{7}{5} = \frac{2 \cdot 7}{3 \cdot 5}$$

Suppose you pause and look carefully at the result you have achieved.

$$\frac{2}{3}\cdot\frac{7}{5}=\frac{2\cdot 7}{3\cdot 5}$$

To find the product of these two particular rational numbers you form a new fraction by using the product of the numerators and the product of the denominators of the fractions you started with. The product $\frac{2\cdot 7}{3\cdot 5}$ is clearly a fraction and hence you know that the product of these two rational numbers is a rational number.

Other numbers could have been used just as well as $\frac{2}{3}$ and $\frac{7}{5}$. The reasoning does not depend upon the particular numbers used. We can use $\frac{a}{b}$ and $\frac{c}{d}$ where a, b, c, and d are any whole numbers with b and d not zero.

$$b\cdot\frac{a}{b}=a \quad\text{and}\quad d\cdot\frac{c}{d}=c$$

$$\left(b\cdot\frac{a}{b}\right)\cdot\left(d\cdot\frac{c}{d}\right)=a\cdot c$$

$$(b\cdot d)\cdot\left(\frac{a}{b}\cdot\frac{c}{d}\right)=a\cdot c$$

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{a\cdot c}{b\cdot d}$$

Product of Two Rational Numbers: If a, b, c, and d are whole numbers with b and d not equal to zero, then

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{a\cdot c}{b\cdot d}$$

This statement tells us a great deal:

1. It tells us that this is the only definition of multiplication of rational numbers which is possible under the requirement which we made. We required that the properties of multiplication listed in Section 6-6 hold when the operation is extended to rational numbers.
2. It tells us how to find a fraction for the product of two rational numbers by using their fraction numerals.
3. It tells us that the system of rational numbers is closed under multiplication. The product of two rational numbers has a fraction numeral and therefore the product is a rational number.

From time to time in our discussion of rational numbers and fractions we shall find it necessary to have names for the a and b of $\frac{a}{b}$. We shall use numerator for a and denominator for b. The context will make clear whether we mean the number or the numeral. We may speak of the product of the numerators and think of them as numbers. We may also speak of the numerators of the fractions and think of them as numerals.

Now tell in your own words how to find a fraction for the product of two rational numbers by using their fraction forms.

Exercises 6-7b

1. Use $\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$ to express the following products as fractions.

   Example: $\frac{4}{5} \cdot \frac{3}{11} = \frac{4 \cdot 3}{5 \cdot 11} = \frac{12}{55}$

   a. $\frac{2}{7} \cdot \frac{5}{9}$
   b. $\frac{1}{3} \cdot \frac{2}{3}$
   c. $\frac{0}{5} \cdot \frac{6}{8}$
   d. $\frac{5}{4} \cdot \frac{6}{5}$
   e. $\frac{1}{6} \cdot \frac{1}{5}$
   f. $\frac{3}{4} \cdot \frac{9}{8}$
   g. $\frac{2}{5} \cdot \frac{5}{3}$
   h. $\frac{2}{5} \cdot \frac{3}{1}$
   i. $\frac{0}{3} \cdot \frac{0}{7}$
   j. $\frac{14}{19} \cdot \frac{92}{60}$
   k. $\frac{14}{60} \cdot \frac{92}{19}$
   l. $\frac{x}{y} \cdot \frac{z}{w}$

2. Express the following products as fractions.

   a. $\frac{3}{4} \cdot \frac{4}{3}$
   b. $\frac{6}{5} \cdot \frac{5}{6}$
   c. $\frac{5}{1} \cdot \frac{1}{5}$
   d. $\frac{4}{7} \cdot \frac{7}{4}$
   e. $\frac{2}{3} \cdot \frac{3}{2}$
   f. $\frac{a}{b} \cdot \frac{b}{a}$

3. Express the following products as fractions.

   a. $\frac{3}{4} \cdot \frac{5}{5}$
   b. $\frac{2}{7} \cdot \frac{3}{3}$
   c. $\frac{9}{2} \cdot \frac{3}{3}$
   d. $\frac{8}{7} \cdot \frac{1}{1}$
   e. $\frac{c}{d} \cdot \frac{1}{1}$

4. Find fractions which make the following statements true.

   a. $\frac{2}{3} \cdot \frac{?}{?} = \frac{10}{21}$

   b. $\frac{4}{5} \cdot \frac{?}{?} = \frac{8}{15}$

   c. $\frac{3}{4} \cdot \frac{?}{?} = \frac{9}{16}$

   d. $\frac{4}{5} \cdot \frac{?}{?} = \frac{4}{5}$

   e. $\frac{6}{5} \cdot \frac{?}{?} = \frac{30}{30}$

   f. $\frac{1}{5} \cdot \frac{?}{?} = \frac{2}{10}$

5. At the West Junior High School, $\frac{3}{7}$ of the teachers were women and $\frac{5}{11}$ of these women teachers were married. What fractional part of all the teachers were married women?

6. A T.V. program lasts $\frac{3}{4}$ of an hour. If $\frac{1}{10}$ of this time is spend on commercials, what fractional part of an hour is spent on commercials?

7. The Men's Shop received a shipment of shoes. Of these, $\frac{8}{15}$ were black shoes. If $\frac{4}{7}$ of the black shoes had rubber soles, what part of the shipment was black rubber-soled shoes?

8. At a certain town election $\frac{5}{8}$ of the population was eligible to vote. If $\frac{5}{7}$ of those eligible to vote actually voted, what fractional part of the town's population actually voted?

Let us review what you know about rational numbers at this stage of your progress.

1. Every rational number has a name which is a fraction.
2. The rational number $\frac{a}{b} (b \neq 0)$ is the number for which $b \cdot \frac{a}{b} = a$.
3. The commutative, associative, distributive properties and the properties of zero and one are retained in the system of rational numbers.
4. The product of two rational numbers is

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d} \quad (a, b, c, d, \text{ are whole numbers } b \neq 0, d \neq 0)$$

What is the product of two rational numbers when one of them is a whole number? For a particular case like $9 \cdot \frac{7}{9}$, which follows the pattern of $b \cdot \frac{a}{b}$, we know that the product is 7. What can we do with

$$10 \cdot \frac{7}{9} \text{ ?}$$

Notice that $10 \cdot \frac{7}{9}$ does not follow the pattern $b \cdot \frac{a}{b}$.

We can make use of item 1 above. Is 10 a rational number? What fraction may we use as a numeral for 10 ? "$\frac{10}{1}$" is a fraction and we may use this numeral to find the product.

We may write $\frac{10}{1} \cdot \frac{7}{9}$.

This is now the product of two rational numbers and we use the pattern in item 4.

$$\frac{10}{1} \cdot \frac{7}{9} = \frac{10 \cdot 7}{1 \cdot 9} = \frac{70}{9}$$

Exercises 6-7c

1. Which property is used in the statement: $10 \cdot \frac{7}{9} = \frac{7}{9} \cdot 10$?
2. Describe in your own words how to write a fraction for the product of two rational numbers when one of them is a whole number.
3. Express the following products as fractions.

   a. $5 \cdot \frac{3}{4}$
   b. $6 \cdot \frac{1}{7}$
   c. $\frac{2}{15} \cdot 4$
   d. $0 \cdot \frac{7}{8}$
   e. $4 \cdot \frac{3}{5}$
   f. $\frac{8}{9} \cdot 11$
   g. $8 \cdot \frac{1}{5}$
   h. $\frac{2}{3} \cdot 4$
   i. $\frac{5}{13} \cdot 0$

4. Express the following rational numbers as the product of a whole number and a fraction with numerator "1". For example, $\frac{5}{3} = 5 \cdot \frac{1}{3}$.

   a. $\frac{7}{6}$
   b. $\frac{2}{5}$
   c. $\frac{3}{4}$
   d. $\frac{4}{9}$
   e. $\frac{3}{3}$
   f. $\frac{6}{9}$

5. At Mountain View School the length of the school day is 6 hours. If $\frac{6}{7}$ of the day is spent in class periods, what is the time spent in class each day?

6-8. Equivalent Fractions.

If you were asked to "simplify" the fraction $\frac{10}{15}$, you would probably say that $\frac{10}{15} = \frac{2}{3}$. This is correct. Why is it correct? It turns out that this fact follows very simply from what you know about the product of two rational numbers.

Let us see how it follows. In what other way can we write $\frac{10}{15}$?

We can write: $\frac{2 \cdot 5}{3 \cdot 5}$. What does $\frac{2 \cdot 5}{3 \cdot 5}$ mean?

We recognize from what we know about the product of two rational numbers that

$$\frac{10}{15} = \frac{2 \cdot 5}{3 \cdot 5} \text{ and } \frac{2 \cdot 5}{3 \cdot 5} = \frac{2}{3} \cdot \frac{5}{5}. \text{ But } \frac{5}{5} = 1.$$

Then $$\frac{2 \cdot 5}{3 \cdot 5} = \frac{2}{3} \cdot 1$$

so that $$\frac{2 \cdot 5}{3 \cdot 5} = \frac{2}{3}$$

or $$\frac{10}{15} = \frac{2}{3}$$

by the property of the multiplication of one, applied to rational numbers.

We speak of "$\frac{10}{15}$" and "$\frac{2}{3}$" as equivalent fractions. They are different fractions representing the same rational number.

We regard $\frac{2}{3}$ as the "simplest form" since the numerator and denominator have no common factors (other than 1). Sometimes it is desirable to find the "simplest form" and sometimes it is desirable to find a particular "equivalent fraction". Suppose we start with the form $\frac{2}{3}$ and wish to find an equivalent fraction in which the denominator is 15. How shall we proceed?

$$\frac{2}{3} = \frac{2}{3} \cdot 1 = \frac{2}{3} \cdot \frac{5}{5} \quad \text{(Multiplication property of 1)}$$

$$= \frac{10}{15}$$

Let us try to work with a symbol which represents any fraction and, in turn, any rational number.

If a, b, and k are whole numbers with b and k not equal to zero, then

$$\frac{a}{b} = \frac{a}{b} \cdot 1 = \frac{a}{b} \cdot \frac{k}{k} = \frac{a \cdot k}{b \cdot k}$$

Then the statement: $\frac{a}{b} = \frac{a \cdot k}{b \cdot k}$

tells us how to write equivalent fractions.

Example 1: Simplify $\frac{18}{42}$

$$\frac{18}{42} = \frac{3 \cdot 6}{7 \cdot 6} = \frac{3}{7} \cdot 1 = \frac{3}{7}$$

Example 2: Express $\frac{2}{5}$ as a fraction in which the denominator is 20.

$$\frac{2}{5} = \frac{2 \cdot 4}{5 \cdot 4} = \frac{8}{20}$$

In this case k is 4 since $5 \cdot 4 = 20$, and 20 is the required denominator.

Exercises 6-8a

(Class Discussion)

1. In Example 1 what numeral in $\frac{3 \cdot 6}{7 \cdot 6}$ corresponds to k in $\frac{a \cdot k}{b \cdot k}$?
2. What values should be given to a, b, and k if you want to simplify $\frac{24}{18}$ to $\frac{4}{3}$?
3. Why is the fraction $\frac{2}{3}$ simpler than the fraction $\frac{10}{15}$? Note that the numeral $\frac{2}{3}$ is simpler but the number $\frac{2}{3}$ is the same as the number $\frac{10}{15}$.

4. Write three different equivalent fractions for each of the fractions below.

a.	$\frac{1}{2}$	f.	$\frac{25}{30}$
b.	$\frac{2}{5}$	g.	$\frac{12}{6}$
c.	$\frac{3}{3}$	h.	$\frac{30}{25}$
d.	$\frac{12}{4}$	i.	$\frac{1}{1}$
e.	$\frac{0}{5}$	j.	$\frac{27}{81}$

5. a. What is the least common multiple of 10 and 15 ?
   b. Use the L.C.M. as the denominator and write equivalent fractions for $\frac{3}{10}$ and $\frac{4}{15}$.

6. Express each of the given rational numbers as a fraction with the indicated denominator.

a.	$\frac{2}{3}$, denom. 15	d.	$\frac{2}{3}$, denom. 111
b.	$\frac{7}{5}$, denom. 15	e.	$\frac{1}{2}$, denom. 18
c.	$\frac{11}{37}$, denom. 111	f.	$\frac{1}{1}$, denom. 18

7. Express the following groups of rational numbers as fractions with the same denominators.

a.	$\frac{2}{3}$, $\frac{7}{5}$	d.	$\frac{11}{37}$, $\frac{2}{3}$
b.	$\frac{3}{10}$, $\frac{4}{15}$	e.	$\frac{1}{4}$, $\frac{1}{5}$
c.	$\frac{2}{7}$, $\frac{5}{6}$		

Exercises 6-8b

1. Simplify the following fractions. Use the method shown in the example.

   Example: $\frac{30}{24} = \frac{2 \cdot 3 \cdot 5}{2 \cdot 3 \cdot 4} = \frac{2}{2} \cdot \frac{3}{3} \cdot \frac{5}{4} = \frac{5}{4}$

   a. $\frac{9}{21}$
   b. $\frac{14}{16}$
   c. $\frac{12}{24}$
   d. $\frac{15}{45}$
   e. $\frac{24}{42}$
   f. $\frac{55}{121}$
   g. $\frac{40}{60}$
   h. $\frac{75}{100}$
   i. $\frac{123}{321}$
   j. $\frac{111}{37}$
   k. $\frac{432}{234}$
   l. $\frac{111111}{1111}$

2. Express each of the given rational numbers as a fraction with the indicated denominator.

   a. 2, denom. 3
   b. 5, denom. 3
   c. $\frac{2}{5}$, denom. 15
   d. $\frac{2}{5}$, denom. 35
   e. $\frac{6}{15}$, denom. 30
   f. $\frac{1}{3}$, denom. 15

3. Express the following groups of rational numbers as fractions with the same denominators.

   a. $\frac{1}{2}$, $\frac{3}{4}$
   b. $\frac{7}{14}$, $\frac{1}{4}$
   c. $\frac{7}{30}$, $\frac{3}{20}$
   d. $\frac{4}{5}$, $\frac{5}{4}$
   e. $\frac{4}{1}$, $\frac{1}{4}$
   f. $\frac{3}{1}$, $\frac{1}{5}$
   g. 3, $\frac{1}{5}$
   h. $\frac{1}{121}$, $\frac{1}{44}$
   i. $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{5}$
   j. $\frac{1}{6}$, $\frac{2}{15}$, $\frac{3}{10}$

4. Multiply and simplify where possible.

   a. $3 \cdot \frac{2}{3}$
   b. $6 \cdot \frac{2}{3}$
   c. $5 \cdot \frac{3}{10}$
   d. $6 \cdot \frac{2}{15}$
   e. $3 \cdot \frac{1}{6}$
   f. $8 \cdot \frac{1}{8}$
   g. $10 \cdot \frac{3}{25}$
   h. $7 \cdot \frac{4}{20}$
   i. $7 \cdot \frac{4}{27}$
   j. $\frac{1}{8} \cdot \frac{7}{8}$
   k. $\frac{3}{4} \cdot \frac{4}{5}$
   l. $\frac{7}{12} \cdot \frac{6}{7}$
   m. $\frac{5}{8} \cdot 5$
   n. $\frac{3}{8} \cdot \frac{5}{6}$
   o. $\frac{9}{16} \cdot 12$

5. A man has 7 children, 3 daughters and the rest sons. He has 20 acres of land to divide equally among all his children. What is the total number of acres that the sons will receive?
6. Hamburger costs 80¢ a pound. How much would you pay for 7 ounces?
7. Walter was paid $173 for 40 hours of work. At this rate, how much will he receive if he works 50 hours?
8. BRAINBUSTER. The numerator and denominator of a fraction contain the same two digits but in opposite order. The fraction is equivalent to $\frac{4}{7}$. Find three such fractions.

6-9. Reciprocals.

Two numbers are said to be reciprocals of each other if the product of the two numbers is 1. What is the reciprocal of 6 ? Another way to ask this question is: What is the number x for which $6 \cdot x = 1$?

$x = \frac{1}{6}$ from the pattern $b \cdot \frac{a}{b} = a$.

Then $6 \cdot \frac{1}{6} = 1$

and 6 and $\frac{1}{6}$ are reciprocals of each other.

The reciprocal of 6 is $\frac{1}{6}$. The reciprocal of $\frac{1}{6}$ is 6.

All rational numbers except 0 have reciprocals. Think about how you can find the reciprocal of $\frac{4}{5}$. The reciprocal of $\frac{4}{5}$ is $\frac{5}{4}$ since

$$\frac{4}{5} \cdot \frac{5}{4} = \frac{4 \cdot 5}{5 \cdot 4}$$

and $\frac{4 \cdot 5}{5 \cdot 4}$ is 1 because the numerator and denominator are the same.

If a and b are whole numbers different from 0, then the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$ since $\frac{a}{b} \cdot \frac{b}{a} = \frac{a \cdot b}{b \cdot a} = 1$.

Exercises 6-9
(Class Discussion)

1. What properties are used in the statement $\frac{a \cdot b}{b \cdot a} = 1$?
2. Describe in your own words an easy way to find the reciprocal of a rational number, written as a fraction.
3. Give the reciprocals of the following numbers.

    a. $\frac{5}{8}$
    b. $\frac{7}{3}$
    c. $\frac{8}{5}$
    d. $\frac{3}{7}$
    e. $\frac{17}{6}$
    f. $\frac{9}{4}$
    g. $\frac{6}{1}$
    h. $\frac{1}{6}$
    i. $\frac{11}{7}$
    j. $\frac{2}{3}$
    k. $\frac{3}{4}$
    l. $\frac{6}{9}$
    m. $\frac{1}{1}$
    n. $\frac{14}{15}$
    o. $\frac{21}{20}$
    p. $\frac{1729}{1492}$

4. Why is there no reciprocal for zero?

6-10. Division of Rational Numbers.

Now we are ready to look at the problem of dividing one rational number by another rational number. When we came to division in our study of counting numbers, we ran into trouble. We noted that we cannot always find a counting number as the quotient when we divide one counting number by another. The set of counting numbers is not closed under division. We had to extend our number system to include rational numbers in order to be able to divide counting numbers.

We now face an interesting question. Is our rational number system closed under division? Can we find a rational number which is the quotient when a rational number is divided by a rational number, or is it necessary to extend the system again?

We wish to investigate $\frac{2}{3} \div \frac{5}{7}$ in order to find whether this expression has any meaning. In our investigation we shall require that the properties listed in Section 6-6 continue to hold true.

We do not yet know anything about division of rational numbers, but we do know about multiplication. We make use of the fact that division is the inverse operation of multiplication.

$$a \div b = x \quad \text{means that} \quad a = b \cdot x$$

We can start by saying that we want to find the number n, if there is such a number, where

$$\frac{2}{3} \div \frac{5}{7} = n\ .$$

By the definition of division as the inverse of multiplication, we have

$$\frac{2}{3} = \frac{5}{7} \cdot n\ .$$

This is helpful because the operation is now multiplication, which we can handle. It would be nice to have just n instead of $\frac{5}{7} \cdot n$. We learned about reciprocals in Section 6-9 and we now find that the product of reciprocals is useful to us.

The reciprocal of $\frac{5}{7}$ is $\frac{7}{5}$ and the product $(\frac{5}{7} \cdot \frac{7}{5})$ is 1.

Let us now proceed:

$\frac{2}{3} = \frac{5}{7} \cdot n$	from above
"$\frac{2}{3}$" and "$\frac{5}{7} \cdot n$"	are different names for the same number.
"$\frac{7}{5} \cdot \frac{2}{3}$" and "$\frac{7}{5} \cdot (\frac{5}{7} \cdot n)$"	are different ways of expressing the product of the same two numbers, $\frac{7}{5}$ and the number named by "$\frac{2}{3}$" and also by "$(\frac{5}{7} \cdot n)$".
$\frac{7}{5} \cdot \frac{2}{3} = \frac{7}{5} \cdot (\frac{5}{7} \cdot n)$	"$\frac{7}{5} \cdot \frac{2}{3}$" and "$\frac{7}{5} \cdot (\frac{5}{7} \cdot n)$" are names for the same number.
$\frac{14}{15} = \frac{7}{5} \cdot (\frac{5}{7} \cdot n)$	$\frac{7}{5} \cdot \frac{2}{3} = \frac{14}{15}$ (product of rational numbers)
$\frac{14}{15} = (\frac{7}{5} \cdot \frac{5}{7}) \cdot n)$	Associative property
$\frac{14}{15} = 1 \cdot n$	$\frac{7}{5}$ and $\frac{5}{7}$ are reciprocals
$\frac{14}{15} = n$	Identity property of 1
Then $\frac{2}{3} \div \frac{5}{7} = n$	Statement we started with
becomes $\frac{2}{3} \div \frac{5}{7} = \frac{14}{15}$	n is $\frac{14}{15}$
or $\frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \cdot \frac{7}{5}$	$\frac{14}{15}$ is $\frac{2}{3} \cdot \frac{7}{5}$ (product of rational numbers)

The last statement says that division by $\frac{5}{7}$ is the same as multiplication by its reciprocal $\frac{7}{5}$. Does division of any rational number by any non-zero rational number follow the same procedure? The answer is yes. Division by any rational number (not zero) is the same as multiplication by its reciprocal.

You have found that $\frac{2}{3} \div \frac{5}{7}$ is $\frac{2}{3} \cdot \frac{7}{5}$ which is a rational number. It is the only possible meaning to attach to $\frac{2}{3} \div \frac{5}{7}$ if the properties stated in Section 6-6 are kept. It shows the procedure to use with fractions in the division of rational numbers and it shows that the rational numbers are closed under division except that division by zero is excluded.

> To find the quotient of two rational numbers written as fractions, find the product of the first rational number and the reciprocal of the second rational number.

Quotient of Two Rational Numbers:

If a, b, c, and d are whole numbers with b and d not equal to zero,

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{a \cdot d}{b \cdot c}$$

When we began our search for n in $\frac{2}{3} \div \frac{5}{7} = n$ we were not sure that such a number existed. We found that indeed there is such a number and that it is $\frac{14}{15}$. Let us check that this number n $(\frac{14}{15})$ really does satisfy

$$\frac{2}{3} = \frac{5}{7} \cdot n$$

$$\frac{5}{7} \cdot n = \frac{5}{7} \cdot \frac{14}{15} = \frac{5}{7} \cdot \frac{7 \cdot 2}{5 \cdot 3} = \frac{5}{7} \cdot (\frac{7}{5} \cdot \frac{2}{3})$$

$$= (\frac{5}{7} \cdot \frac{7}{5}) \cdot \frac{2}{3} = 1 \cdot \frac{2}{3} = \frac{2}{3} .$$

Exercises 6-10a

Express each of the following as a fraction:

	(a)	(b)	(c)
1.	$\frac{2}{3} + \frac{5}{7}$	$\frac{2}{5} + \frac{3}{7}$	$\frac{2}{7} + \frac{5}{3}$
2.	$\frac{5}{7} + \frac{2}{3}$	$\frac{3}{7} + \frac{2}{5}$	$\frac{5}{3} + \frac{2}{7}$
3.	$\frac{3}{2} + \frac{5}{7}$	$\frac{5}{2} + \frac{3}{7}$	$\frac{7}{2} + \frac{5}{3}$
4.	$\frac{5}{7} + \frac{3}{2}$	$\frac{3}{7} + \frac{5}{2}$	$\frac{5}{3} + \frac{7}{2}$
5.	$\frac{2}{3} + \frac{7}{5}$	$\frac{2}{5} + \frac{7}{3}$	$\frac{2}{7} + \frac{3}{5}$
6.	$\frac{7}{5} + \frac{2}{3}$	$\frac{7}{3} + \frac{2}{5}$	$\frac{3}{5} + \frac{2}{7}$
7.	$\frac{3}{2} + \frac{7}{5}$	$\frac{5}{2} + \frac{7}{3}$	$\frac{7}{2} + \frac{3}{5}$
8.	$\frac{7}{5} + \frac{3}{2}$	$\frac{7}{3} + \frac{5}{2}$	$\frac{3}{5} + \frac{7}{2}$

Express each of the following as a fraction in simplest form:

9.	$\frac{2}{3} + \frac{5}{3}$	$\frac{1}{2} + \frac{1}{4}$	$\frac{25}{4} + \frac{5}{8}$
10.	$\frac{3}{4} + \frac{7}{4}$	$\frac{1}{6} + \frac{1}{3}$	$\frac{14}{15} + \frac{2}{5}$
11.	$\frac{3}{7} + \frac{4}{7}$	$\frac{3}{4} + \frac{3}{4}$	$\frac{10}{11} + \frac{12}{13}$
12.	$\frac{5}{7} + \frac{5}{8}$	$\frac{15}{16} + \frac{5}{4}$	$\frac{5}{6} + \frac{7}{8}$

13. Mrs. Brown has $\frac{15}{4}$ pounds of potato salad. How many portions of $\frac{1}{8}$ pound can she serve?

14. A metal worker has a steel bar $\frac{3}{4}$ of a foot long. How many pieces $\frac{1}{8}$ of a foot long can be cut from the bar?

15. A wire fence extends over a distance of $\frac{3}{8}$ of a mile. At each $\frac{1}{64}$ of a mile a wooden post is used as a support. How many wooden posts are used? (Don't forget to count the posts at each end.)

Exercises 6-10b

1. Write the reciprocals of the following rational numbers:
   - a. 11
   - b. 201
   - c. 47
   - d. $\frac{1}{5}$
   - e. $\frac{2}{7}$
   - f. $\frac{50}{3}$
   - g. $\frac{1000}{7}$
   - h. $\frac{346}{175}$

2. In the following, letters represent rational numbers, all different from zero. Write the reciprocals.

   a. m b. s c. $\frac{1}{c}$ d. $\frac{r}{s}$ e. $\frac{t}{w}$

3. Write the set of numbers consisting of the reciprocals of the members of the set Q where:

$$Q = \{1, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}\}.$$

4. Find the value of x so that the following statements are true:
   - a. $3 \cdot x = 1$
   - b. $\frac{1}{7} \cdot x = 1$
   - c. $\frac{2}{3} \cdot x = 1$
   - d. $\frac{7}{5} \cdot x = 1$
   - e. $100 \cdot x = 1$
   - f. $\frac{100}{3} \cdot x = 1$
   - g. $2 \cdot x = 5$

Express each of the following in simplest form:

5. $\frac{2}{3} \cdot \frac{2}{3}$
6. $3 \cdot \frac{7}{3}$
7. $\frac{7}{4} + \frac{7}{4}$
8. $\frac{1}{2} + 1$
9. $9 \cdot \frac{0}{3}$
10. $\frac{3}{4} + \frac{3}{4}$
11. $\frac{15}{7} \cdot 7$
12. $\frac{1}{7} + \frac{1}{2}$
13. $\frac{22}{3} + \frac{5}{3}$
14. $\frac{4}{3} + \frac{5}{4}$
15. $\frac{11}{3} + \frac{7}{4}$
16. $\frac{1}{2} + \frac{2}{3}$
17. $\frac{1}{2} + \frac{1}{3}$
18. $\frac{2}{1} + \frac{3}{5}$
19. $\frac{6}{7} + \frac{5}{3}$
20. $\frac{9}{5} + \frac{2}{1}$
21. $\frac{4}{5} + \frac{5}{6}$

Exercises 6-10c

1. Express each of the following as a fraction in simplest form:

   a. $\frac{1}{2} \div \frac{1}{2}$

   b. $12 \div \frac{1}{6}$

   c. $\frac{5}{9} \div \frac{9}{5}$

   d. $\frac{5}{8} \div \frac{5}{4}$

   e. $\frac{11}{24} \div \frac{11}{2}$

   f. $\frac{1}{2} \div \frac{1}{3}$

2. Do you think division is commutative? Express each quotient as a fraction to test your response.

   a. $\frac{2}{3} \div \frac{1}{2}$

   b. $\frac{1}{2} \div \frac{2}{3}$

3. Do you think that division is associative? Express each quotient as a fraction to test your response.

   a. $\frac{3}{2} \div (\frac{9}{4} \div \frac{7}{6})$

   b. $(\frac{3}{2} \div \frac{9}{4}) \div \frac{7}{6}$

4. A large can of orange juice is $\frac{7}{8}$ full. If a portion of orange juice is $\frac{1}{16}$ of this large can, how many people can be served by using the juice in the can?

5. How many meat patties can be made from $\frac{3}{4}$ pound of meat if each patty is to contain $\frac{1}{8}$ pound of meat?

6. Mrs. Edwards bought $\frac{1}{2}$ of a watermelon. She cut the watermelon into 8 equal portions. What part of a whole melon is each portion?

6-11. Addition of Rational Numbers.

It is time to pretend again. This time we shall investigate addition of rational numbers. We shall use the distributive property. This seems reasonable since this property uses multiplication which we know about and addition which we wish to study.

Our problem is to extend addition to rational numbers and to keep the properties of addition. We shall start with the simplest kind of addition example where the fractions have the same denominator.

Example 1: Add $\frac{2}{7}$ and $\frac{3}{7}$

We know that $2 \cdot \frac{1}{7} = \frac{2}{7}$ Product of two rational numbers.

and $3 \cdot \frac{1}{7} = \frac{3}{7}$

$$\frac{2}{7} + \frac{3}{7} = 2 \cdot \frac{1}{7} + 3 \cdot \frac{1}{7}$$

$$\frac{2}{7} + \frac{3}{7} = (2 + 3) \cdot \frac{1}{7} \qquad \text{Distributive property}$$

$$\frac{2}{7} + \frac{3}{7} = 5 \cdot \frac{1}{7} \qquad 2 + 3 = 5$$

$$\frac{2}{7} + \frac{3}{7} = \frac{5}{7} \qquad 5 \cdot \frac{1}{7} = \frac{5}{7}$$

Example 2: Use $\frac{a}{b}$ and $\frac{c}{b}$ $(b \neq 0)$ to represent two rational numbers whose fractions have the same denominator.

Add $\frac{a}{b} + \frac{c}{b}$

We know that $\frac{a}{b}$ is $a \cdot \frac{1}{b}$

and $\frac{c}{b}$ is $c \cdot \frac{1}{b}$

$$\frac{a}{b} + \frac{c}{b} = a \cdot \frac{1}{b} + c \cdot \frac{1}{b}$$

$$\frac{a}{b} + \frac{c}{b} = (a + c) \cdot \frac{1}{b}$$

$$\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}$$

Exercises 6-11a
(Class Discussion)

1. Describe in your own words the procedure to follow to find the sum of two rational numbers in fraction form in which the denominators are the same.
2. What fraction represents the sum of $\frac{p}{s} + \frac{q}{s}$ where p, q, and s are whole numbers, and $s \neq 0$?

3. **Express as fractions:**

a. $\frac{1}{5} + \frac{3}{5}$

b. $\frac{3}{10} + \frac{4}{10}$

c. $\frac{1}{3} + \frac{1}{3}$

d. $\frac{2}{9} + \frac{5}{9}$

e. $\frac{2}{7} + \frac{6}{7}$

f. $\frac{3}{8} + \frac{6}{8}$

g. $\frac{4}{3} + \frac{7}{3}$

h. $\frac{2}{9} + \frac{3}{9}$

4. **Express as fractions in simplest form:**

a. $\frac{2}{5} + \frac{3}{5}$

b. $\frac{3}{10} + \frac{5}{10}$

c. $\frac{3}{100} + \frac{5}{100}$

d. $\frac{3}{10} + \frac{7}{10}$

e. $\frac{2}{9} + \frac{4}{9}$

f. $\frac{1}{8} + \frac{5}{8}$

g. $\frac{4}{7} + \frac{3}{7}$

h. $\frac{11}{7} + \frac{10}{7}$

5. **Find the least common multiple of the denominators in each pair of fractions.**

a. $\frac{3}{10}, \frac{4}{15}$

b. $\frac{2}{9}, \frac{7}{6}$

c. $\frac{7}{8}, \frac{11}{4}$

d. $\frac{17}{12}, \frac{2}{5}$

e. $\frac{7}{3}, \frac{15}{4}$

f. $\frac{29}{50}, \frac{3}{20}$

g. $\frac{23}{12}, \frac{9}{8}$

h. $\frac{7}{6}, \frac{2}{15}$

6. Use the least common multiples of the pairs of denominators in Problem 5 to write equivalent fractions with the same denominator for each pair of fractions.

Example: a. $\frac{3}{10} = \frac{3}{10} \cdot \frac{3}{3} = \frac{9}{30}$

$\frac{4}{15} = \frac{4}{15} \cdot \frac{2}{2} = \frac{8}{30}$

So far, you have learned how to add rational numbers of a particular kind. The fraction forms of these rational numbers have the same denominator. What can be done when the fractions have different denominators? Simply find equivalent fractions in which the denominators are the same.

Consider the problem of finding a fraction for the sum:

$$\frac{2}{5} + \frac{3}{7}$$

Any common multiple of 5 and 7 will serve for a denominator, but the least common multiple is usually preferred.

What is the least common multiple of 5 and 7 ? Write equivalent fractions and add:

$$\frac{2}{5} = \frac{14}{35} \text{ and } \frac{3}{7} = \frac{15}{35}$$

$$\frac{2}{5} + \frac{3}{7} = \frac{14}{35} + \frac{15}{35} = \frac{29}{35}$$

Exercises 6-11b

1. Express each sum as a fraction and simplify where it is possible.

   a. $\frac{2}{5} + \frac{3}{7}$

   b. $\frac{2}{5} + \frac{2}{7}$

   c. $\frac{1}{2} + \frac{1}{3}$

   d. $\frac{3}{10} + \frac{2}{15}$

   e. $\frac{5}{6} + \frac{3}{14}$

   f. $\frac{4}{3} + \frac{3}{4}$

   g. $\frac{4}{15} + \frac{3}{20}$

   h. $\frac{1}{6} + \frac{1}{3}$

   i. $\frac{3}{4} + \frac{1}{2}$

   j. $\frac{1}{2} + \frac{1}{6}$

   k. $\frac{5}{1} + \frac{1}{3}$

   l. $\frac{1}{33} + \frac{3}{22}$

2. If a, b, c, and d are whole numbers with b and d not equal to zero, find a fraction for $\frac{a}{b} + \frac{c}{d}$.

   a. What is a common multiple of b and d?

   b. $\frac{a}{b} = \frac{? \cdot ?}{b \cdot d}$

   c. $\frac{c}{d} = \frac{? \cdot ?}{b \cdot d}$

   d. $\frac{a \cdot d}{b \cdot d} + \frac{b \cdot c}{b \cdot d} = \frac{?}{?}$

3. Express each sum as a fraction by finding least common denominators. Simplify where possible.

   a. $\frac{3}{10} + \frac{2}{5}$

   b. $\frac{2}{5} + \frac{3}{7}$

   c. $\frac{5}{6} + \frac{3}{14}$

   d. $\frac{4}{15} + \frac{3}{20}$

   e. $\frac{4}{3} + \frac{3}{4}$

   f. $\frac{1}{6} + \frac{1}{3}$

   g. $\frac{3}{4} + \frac{1}{2}$

   h. $\frac{1}{2} + \frac{1}{6}$

   i. $\frac{1}{6} + \frac{2}{9}$

   j. $\frac{1}{6} + \frac{7}{9}$

   k. $\frac{4}{33} + \frac{3}{22}$

   l. $\frac{1}{33} + \frac{3}{22}$

   m. $\frac{2}{15} + \frac{2}{25}$

   n. $5 + \frac{7}{6}$

   o. $4 + \frac{2}{9}$

4. Express each of the following as a fraction in simplest form.

   a. $(\frac{1}{3} + \frac{2}{3}) + \frac{4}{3}$

   b. $\frac{1}{3} + (\frac{2}{3} + \frac{4}{3})$

   c. $(\frac{1}{4} + \frac{5}{4}) + \frac{3}{8}$

   d. $\frac{1}{4} + (\frac{5}{4} + \frac{3}{8})$

5. In a recipe the ingredients include $\frac{1}{3}$ lb. butter, $\frac{1}{2}$ lb. sugar, $\frac{1}{8}$ lb. cocoa, and $\frac{1}{6}$ lb. peanuts. What is the total weight of these ingredients?

6. It is $\frac{1}{6}$ of a mile from Ben's home to school and $\frac{3}{8}$ of a mile from his home to Lincoln Park. How far does Ben walk when he goes home from school and then on to Lincoln Park?

7. The White family spend $\frac{1}{4}$ of their income for rent and $\frac{1}{3}$ of their income for food.

   a. What fractional part of their income is spent on both rent and food?

   b. What fractional part of their income is left over?

8. On a long motor trip Mr. Downs covered $\frac{1}{3}$ of the distance the first day, $\frac{1}{5}$ of the distance the second day, and $\frac{3}{10}$ of the distance the third day.

   a. What part of the trip did Mr. Downs cover during the first three days?

   b. What part of the trip did Mr. Downs still have to cover?

9. A man spends $\frac{1}{5}$ of his salary for rent, $\frac{1}{3}$ for food, $\frac{1}{6}$ for clothing, and $\frac{1}{4}$ for charity and service. The rest he saves. Out of each dollar he earns, how much does he save?

10. In the magic square below, add the numbers in each column. Then, adding across, find the sum of the numbers in each row. Now add the numbers in each diagonal. (Top left corner to lower right corner, etc.)

$\frac{45}{16}$	$\frac{5}{8}$	$\frac{35}{16}$
$\frac{5}{4}$	$\frac{15}{8}$	$\frac{10}{4}$
$\frac{25}{16}$	$\frac{25}{8}$	$\frac{15}{16}$

Sum of Two Rational Numbers: If a, b, c, and d are whole numbers with b and d not equal to zero, then

$$\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}$$

$$\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d + b \cdot c}{b \cdot d}$$

The statement above tells us that the rational numbers are closed with respect to addition and describes the procedure to find a fraction for the sum of two rational numbers.

This is the only possible definition of addition of rational numbers consistent with the properties of addition extended to rational numbers.

6-12. Subtraction of Rational Numbers.

Consider the difference $\frac{7}{9} - \frac{2}{9}$. Subtraction is the inverse of addition which we know how to do. The difference, $\frac{7}{9} - \frac{2}{9}$, is the number which we add to $\frac{2}{9}$ to get $\frac{7}{9}$.

Then $\frac{7}{9} - \frac{2}{9}$ is the number $\frac{n}{9}$ for which

$\frac{2}{9} + \frac{n}{9} = \frac{7}{9}$. Why was the form $\frac{n}{9}$ chosen here?

It is easy to see that $\frac{2}{9} + \frac{5}{9} = \frac{7}{9}$.

Hence $\frac{7}{9} - \frac{2}{9}$ is the number $\frac{5}{9}$.

$\frac{7}{9} - \frac{2}{9} = \frac{5}{9}$ can be expressed as $\frac{7}{9} - \frac{2}{9} = \frac{7-2}{9}$.

The procedure for subtraction is

$$\frac{7}{9} - \frac{2}{9} = \frac{7-2}{9} = \frac{5}{9}.$$

We often do the middle step mentally. We check by addition:

$$\frac{2}{9} + \frac{5}{9} = \frac{7}{9}.$$

Subtraction of Rational Numbers: If $\frac{a}{b}$ and $\frac{c}{b}$ where $b \neq 0$ are two rational numbers, then

$$\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}.$$

Exercises 6-12a

1. Express the following as fractions.

   a. $\frac{17}{14} - \frac{11}{14}$
   b. $\frac{32}{7} - \frac{15}{7}$
   c. $\frac{11}{5} - \frac{9}{5}$
   d. $\frac{7}{9} - \frac{2}{9}$
   e. $\frac{7}{9} - \frac{5}{9}$
   f. $\frac{17}{11} - \frac{6}{11}$
   g. $\frac{14}{15} - \frac{14}{15}$
   h. $\frac{3}{4} - \frac{1}{4}$
   i. $\frac{12}{3} - \frac{2}{3}$

2. Express the following as fractions and check by addition.

   a. $\frac{9}{11} - \frac{7}{11}$
   b. $\frac{53}{5} - \frac{14}{5}$
   c. $\frac{3}{7} - \frac{1}{7}$
   d. $\frac{4}{3} - \frac{4}{3}$
   e. $\frac{17}{14} - \frac{11}{14}$
   f. $\frac{17}{14} - \frac{3}{14}$
   g. $\frac{3}{6} - \frac{2}{6}$
   h. $\frac{3}{6} - \frac{1}{6}$
   i. $\frac{8}{20} - \frac{5}{20}$

3. a. How do you proceed when the denominators are different as in $\frac{7}{8} - \frac{3}{5}$?

   b. What can you use for a common denominator? What is the least common denominator?

c. What two fractions equivalent to $\frac{7}{8}$ and $\frac{3}{5}$ have the denominator 40 ?

d. What is the simplest fraction for $\frac{7}{8} - \frac{3}{5}$?

4. Express the numbers as fractions with the same denominator and write a single fraction for each difference.

a. $\frac{7}{8} - \frac{3}{5}$
b. $\frac{1}{2} - \frac{1}{3}$
c. $\frac{1}{2} - \frac{1}{6}$
d. $\frac{2}{5} - \frac{1}{4}$
e. $\frac{2}{3} - \frac{4}{9}$
f. $\frac{2}{3} - \frac{2}{9}$
g. $2 - \frac{3}{2}$
h. $7 - \frac{13}{3}$
i. $\frac{14}{5} - 2$
j. $\frac{17}{9} - 1$
k. $\frac{2}{3} - \frac{3}{5}$
l. $\frac{17}{7} - \frac{19}{9}$

5. Express each of the following as a fraction and simplify when it is possible.

a. $\frac{7}{15} - \frac{3}{10}$
b. $\frac{7}{20} - \frac{1}{12}$
c. $\frac{17}{10} - \frac{5}{6}$
d. $\frac{7}{15} - \frac{2}{21}$
e. $\frac{7}{22} - \frac{3}{10}$
f. $\frac{7}{10} - \frac{3}{14}$
g. $\frac{7}{15} - \frac{1}{6}$
h. $\frac{7}{6} - \frac{5}{14}$
i. $\frac{13}{35} - \frac{3}{10}$
j. $\frac{5}{14} - \frac{4}{21}$
k. $\frac{9}{14} - \frac{12}{35}$
l. $\frac{23}{35} - \frac{11}{21}$

*6. Find a fraction for $\frac{a}{b} - \frac{c}{d}$ where a, b, c, and d are whole numbers and $b \neq 0$, $d \neq 0$.

a. Express $\frac{a}{b}$ and $\frac{c}{d}$ as fractions with the same denominator.

b. $\frac{a}{b} - \frac{c}{d} = \frac{?}{?}$

Exercises 6-12b

Express each of the following as fractions and simplify when it is possible.

1. $\frac{6}{7} - \frac{5}{6}$
2. $\frac{17}{3} - \frac{11}{2}$
3. $\frac{15}{2} - \frac{4}{3}$
4. $\frac{11}{5} - \frac{2}{3}$
5. $\frac{5}{6} - \frac{3}{10}$
6. $\frac{11}{10} - \frac{14}{15}$
7. $\frac{5}{6} - \frac{9}{14}$
8. $\frac{1}{3} - \frac{2}{9}$
9. $\frac{7}{11} - \frac{7}{12}$

10. During the first hour of travel a plane covered $\frac{3}{8}$ of the whole trip. At the end of the second hour $\frac{15}{16}$ of the trip had been completed. What part of the trip was covered during the second hour?

11. Over a ten-year period the city of Spring Falls grew in population from $\frac{1}{12}$ of a million to $\frac{1}{9}$ of a million. By what fraction part of a million did Spring Falls grow in this ten-year period?

12. Betty had a piece of ribbon $\frac{5}{6}$ of a yard long. She used $\frac{1}{4}$ of a yard to make a bow. How much ribbon was left? Was this more or less than $\frac{1}{2}$ a yard?

6-13. Summary.

In this chapter you saw why it was necessary to extend the number system from the whole numbers to the rational numbers. This extension was necessary because the counting numbers are not closed under division. Need for the rational numbers arose when man started to use numbers to measure.

Fractions are names for rational numbers. The fractions "$\frac{4}{3}$", "$\frac{8}{6}$", "$\frac{20}{15}$" are all names for the same number.

The set of rational numbers is closed under the operations of addition, multiplication and division. You have learned certain procedures for expressing in the form of a fraction the sum, difference, product and quotient of two rational numbers. These procedures are not picked out of a hat. They follow logically from the basic properties of the operations of multiplication and addition. The basic procedures of operation with rational numbers are given below.

If a, b, c and d are whole numbers with b and d not equal to 0 then:

Product: $\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$

Equivalent fractions: $\frac{a \cdot d}{b \cdot d} = \frac{a}{b}$

Division: $\frac{a}{b} \div \frac{c}{d} = \frac{a \cdot d}{b \cdot c}$ (c not zero)

Addition: $\begin{cases} \frac{a}{b} + \frac{c}{b} = \frac{a + c}{b} \\ \frac{a}{b} + \frac{c}{d} = \frac{a \cdot d + b \cdot c}{b \cdot d} \end{cases}$

Subtraction: $\begin{cases} \frac{a}{b} - \frac{c}{b} = \frac{a - c}{b} \\ \frac{a}{b} - \frac{c}{d} = \frac{a \cdot d - b \cdot c}{b \cdot d} \end{cases}$

Definition: $b \cdot \frac{a}{b} = a$ where $\frac{a}{b}$ is a rational number.

Reciprocals: Two numbers are reciprocals of each other if the product of the two numbers is one.

6-14. Chapter Review.

Exercises 6-14

1. Copy and complete or find the correct value of x.

a. $2 \cdot x = 6$

b. $? \cdot \frac{4}{3} = 4$

c. $7 \cdot \frac{5}{7} = ?$

d. $? \cdot \frac{5}{4} = ?$

e. $6 \cdot \frac{?}{?} = 5$

f. $4 \cdot x = 3$

2. Multiply and simplify.

a. $\frac{1}{2} \cdot \frac{2}{3}$

b. $\frac{2}{5} \cdot \frac{5}{4}$

c. $\frac{7}{5} \cdot \frac{10}{5}$

d. $\frac{12}{3} \cdot \frac{1}{5}$

e. $\frac{2}{5} \cdot \frac{3}{7}$

f. $5 \cdot \frac{2}{9}$

g. $2 \cdot \frac{4}{7}$

h. $12 \cdot \frac{2}{3}$

i. $\frac{22}{7} \cdot \frac{31}{10}$

j. $\frac{20}{3} \cdot \frac{6}{25}$

k. $\frac{1}{3} \cdot (\frac{2}{5} \cdot \frac{1}{2})$

l. $\frac{1}{2} \cdot (\frac{3}{7} \cdot \frac{14}{3})$

m. $\frac{3}{4} \cdot (\frac{2}{5} \cdot \frac{10}{3})$

n. $(\frac{4}{5} \cdot \frac{2}{7}) \cdot \frac{35}{9}$

o. $\frac{1}{2} \cdot (\frac{2}{3} \cdot \frac{3}{4})$

3. Divide and simplify.

a. $\frac{2}{3} \div \frac{4}{3}$

b. $\frac{4}{7} \div \frac{4}{7}$

c. $\frac{4}{7} \div \frac{7}{4}$

d. $\frac{5}{9} \div \frac{2}{3}$

e. $\frac{6}{4} \div \frac{5}{2}$

f. $\frac{1}{2} \div \frac{2}{3}$

g. $\frac{2}{3} \div \frac{3}{4}$

h. $\frac{3}{4} \div \frac{4}{5}$

i. $\frac{14}{15} \div \frac{21}{10}$

j. $\frac{22}{15} \div \frac{11}{5}$

k. $\frac{9}{4} \div \frac{3}{2}$

l. $\frac{1}{8} \div \frac{1}{3}$

m. $5 \div \frac{2}{7}$

n. $\frac{3}{8} \div 5$

o. $\frac{5}{9} \div 4$

p. $14 \div \frac{7}{8}$

4. Add and simplify.

a. $\frac{1}{2} + \frac{1}{3}$

b. $\frac{1}{3} + \frac{1}{4}$

c. $\frac{7}{15} + \frac{3}{10}$

d. $\frac{5}{6} + \frac{3}{10}$

e. $\frac{2}{7} + \frac{1}{3}$

f. $\frac{3}{8} + \frac{1}{4}$

g. $(\frac{5}{12} + \frac{2}{3}) + \frac{1}{6}$

h. $(\frac{3}{2} + \frac{2}{5}) + \frac{1}{3}$

i. $(\frac{1}{2} + \frac{1}{4}) + \frac{1}{8}$

j. $\frac{2}{1} + \frac{5}{6}$

k. $2 + \frac{5}{6}$

l. $\frac{3}{4} + 5$

5. Subtract and simplify.

a. $\frac{5}{6} - \frac{1}{2}$

b. $\frac{5}{6} - \frac{1}{3}$

c. $\frac{10}{21} - \frac{5}{14}$

d. $\frac{7}{3} - 2$

e. $\frac{5}{9} - \frac{1}{3}$

f. $\frac{7}{6} - \frac{6}{7}$

g. $\frac{3}{2} - \frac{2}{3}$

h. $\frac{7}{10} - \frac{4}{15}$

i. $\frac{20}{3} - \frac{3}{4}$

j. $\frac{16}{5} - 3$

k. $5 - \frac{11}{3}$

l. $\frac{11}{9} - \frac{9}{11}$

6. Perform the indicated operations and simplify.

a. $\frac{6}{5} \cdot \frac{5}{6}$

b. $\frac{4}{15} + \frac{2}{5}$

c. $\frac{4}{15} \div 5$

d. $\frac{5}{14} + \frac{2}{7}$

e. $\frac{5}{14} \div \frac{2}{7}$

f. $\frac{2}{9} \cdot \frac{3}{4}$

g. $\frac{3}{10} \div 2$

h. $\frac{3}{10} \cdot 2$

i. $\frac{5}{6} + \frac{3}{4}$

j. $\frac{5}{6} - \frac{3}{4}$

k. $\frac{23}{14} - \frac{11}{7}$

l. $\frac{24}{25} \cdot \frac{5}{6}$

m. $\frac{2}{3} + \frac{3}{4}$

n. $\frac{14}{15} \div \frac{7}{5}$

o. $\frac{13}{7} - \frac{5}{6}$

7. A piece of plywood is made of three plies of thickness $\frac{1}{8}"$, $\frac{3}{16}"$, $\frac{5}{32}"$. How thick is the board?

8. Two-thirds of a cake is divided into 3 pieces. How much of the cake is each piece?

9. Before Edward moved, his home was $\frac{3}{4}$ mile from his school. His new house is 2 blocks closer to the school. (Assume that a block is $\frac{1}{10}$ of a mile.) How far from school is his new house?

10. A recipe calls for $\frac{3}{8}$ of a cup of milk. If Mrs. Cook reduces the recipe by $\frac{1}{3}$, how much milk should she use?

11. Four boys went on a picnic. In their lunch was half of a watermelon. They divided it equally. What part of the whole melon did each boy get?

12. If $\frac{3}{8}$ of the eighth grade class usually works during the summer, how many members of a class of 304 will probably work during the summer?

13. How many quarts of milk are needed by a family of 6 persons for a week if each person uses $\frac{3}{4}$ of a quart each day?

14. What property of numbers is illustrated by $\frac{3}{7} \cdot \frac{2}{2} = \frac{3}{7}$?

15. What is the least common multiple of the denominators of $\frac{5}{6}$ and $\frac{1}{10}$?

16. If the product of two numbers is 1, what are the numbers called?

17. How can you decide whether two fractions name the same rational number?

Complete the following statements:

18. $\frac{2}{3}$ is the number which when multiplied by $\underline{\ ?\ }$ is $\underline{\ ?\ }$.

19. $\frac{2}{3}$ is the product of $?$ by $\frac{1}{?}$.

20. $\frac{2}{3} \cdot ? = 1$.

6-15. Cumulative Review.

Exercises 6-15

1. Find another name for $\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3}$
   a. as a fraction.
   b. using an exponent.

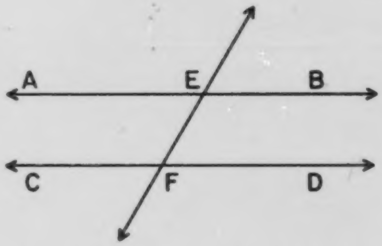
2. a. What are common names for 5^3 and 3^5 ?
   b. Which is larger and by how much?

3. Use set notation to describe the set of all multiples of 10. Is this set closed under addition?

4. What whole numbers can you replace n by so that the following statements are true?
   a. $8 + n = 15$
   b. $8 \cdot n = 15$
   c. $8 + n < 15$

5. True or False. If the intersection of two lines is exactly one point, then the two lines lie in the same plane. Give a reason for your answer.

6. In the diagram there are three lines. $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel.
   a. What is $\overleftrightarrow{EF} \cap \overleftrightarrow{AB}$?
   b. What is $\overleftrightarrow{EF} \cap \overleftrightarrow{CD}$?
   c. What is $\overleftrightarrow{AB} \cap \overleftrightarrow{CD}$?



7. How many different prime factors does 100 have?

8. Can you tell if:
   a. 10101 is divisible by 3 without dividing?
   b. Is 10101 divisible by 9 ?

9. What is the least common multiple of 2, 4, and 10 ?

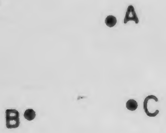
10. If a number has 10 as a factor, what other factors must the number have?

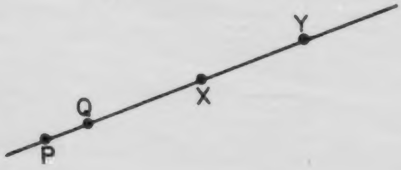
Chapter 7

NON-METRIC GEOMETRY II

7-1. Segments.

Consider three points A, B, and C as in the figure below. Do we say that any one of them is between the other two? No, we do not.





We use the word "between" only when the points in question are on the same line. Look at points P, Q, X and Y above on the line $\overleftrightarrow{PY}$.

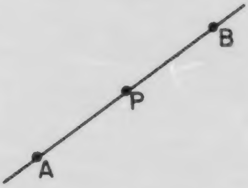
Is X between Q and Y?
Is Q between P and Y?
Is P between X and Y?

Your answers should be Yes, Yes, and No.

We know that while Q is between P and X, there are many other points between P and X. These other points have not been labeled.

When we say that a point P is between points A and B, we mean two things:

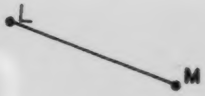
1. There is a line containing A, B, and P.
2. On that line, P is between A and B.



Look at the line $\overleftrightarrow{AB}$ again. Are there points other than P between A and B? We have not labeled any but we know that there are many points between A and B.

We can now say what we mean by segment. Think of two different points L and M. The set of points consisting of

L, M, and all points between L and M is called the segment $\overline{LM}$. Points L and M are called the endpoints of the segment. We name the segment which has endpoints L and M by $\overline{LM}$ or by $\overline{ML}$. A segment is a part of a line.



In the figure below, we can name segments $\overline{AB}$, $\overline{DE}$, and $\overline{AC}$. Can you name other segments?

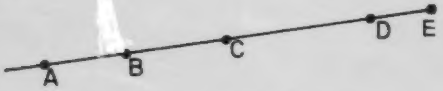


Figure 1

You have studied about the intersection of sets. Do you agree that in the above figure the intersection of segment $\overline{AC}$ and segment $\overline{BD}$ is segment $\overline{BC}$? ($\overline{AC} \cap \overline{BD} = \overline{BC}$)

Exercises 7-1a

1. In Figure 2:
   a. Name two segments whose intersection is point C. Name the endpoints of each segment.
   b. Name three segments whose intersection is point B. Name the endpoint of each segment.

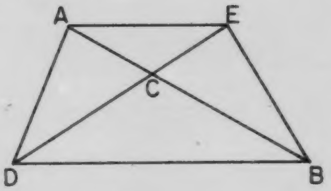


Figure 2

   c. Explain the difference between $\overleftrightarrow{AD}$ and $\overline{AD}$.
   d. What is $\overline{AB} \cap \overline{DE}$?
   e. What is $\overline{AD} \cap \overline{EB}$?

2. In Figure 3:
   a. Name 4 segments that intersect at Z.
   b. What is $\overline{ZY} \cap \overline{XV}$?
   c. What is $\overline{VW} \cap \overline{VZ}$?
   d. What is $\overline{XZ} \cap \overline{YV}$?
   e. What is $\overline{XV} \cap \overline{XY}$?

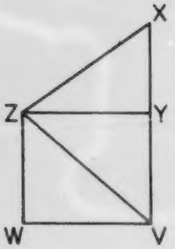


Figure 3

3. In Figure 4:
   a. Name 3 segments on $\overleftrightarrow{TF}$.
   b. What is $\overline{EM} \cap \overline{DM}$?
   c. How many points are there between T and F? Name two of these points.
   d. What is $\overline{TK} \cap \overline{DF}$?
   e. Name at least 6 segments on $\overleftrightarrow{AL}$.

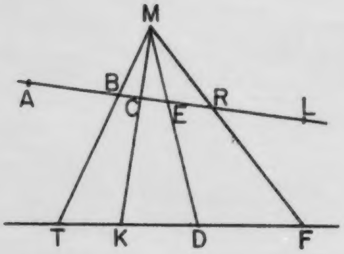


Figure 4

4. Draw a segment. Label its endpoints X and Y.
   a. Is there a pair of points of $\overline{XY}$ with Y between them?
   b. Is there a pair of points of $\overleftrightarrow{XY}$ with Y between them?

5. Draw two segments $\overline{AB}$ and $\overline{CD}$ for which $\overline{AB} \cap \overline{CD}$ is empty but $\overleftrightarrow{AB} \cap \overleftrightarrow{CD}$ is one point.

6. Draw two segments $\overline{PQ}$ and $\overline{RS}$ for which $\overline{PQ} \cap \overline{RS}$ is empty but $\overleftrightarrow{PQ}$ is $\overleftrightarrow{RS}$.

7. In Figure 5:
   a. Is $\overline{AB} \cap \ell$ empty? Explain.
   b. Is $\overline{AC} \cap \ell$ empty? Explain.

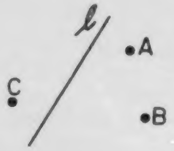


Figure 5

Union of Sets

You may remember that the intersection of two sets:

$$A = \{2,\ 4,\ 6,\ 8,\ 10\}$$

$$B = \{5,\ 6,\ 7,\ 8,\ 9,\ 10\}$$

is a set

$$C = \{6,\ 8,\ 10\},$$

whose members are in both set A and set B.

It is sometimes helpful to talk about a different set formed from set A and set B. Suppose we think about the set

$$D = \{2,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10\}.$$

The set D includes all the members of A and all the members of B. Set D is called the union of sets A and B. Notice that members 6, 8, and 10 appear in both sets A and B. We agree to list these elements only once in the union. The symbol used to show union is $\cup$. We read:

$$A \cup B = D$$

as "the union of set A and set B is the set D."

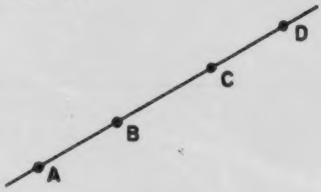
Another example of the union of two sets is given below:

Let set P be the set of girls in your mathematics class.

Let set Q be the set of boys in your mathematics class.

Then $P \cup Q$ is the set of all the students in your mathematics class.

A geometrical example can be seen in the picture at the right. Here $\overline{AB} \cup \overline{BC} = \overline{AC}$ and $\overline{AC} \cup \overline{BD} = \overline{AD}$ and $\overline{AC} \cup \overline{CD} = \overline{AD}$.



Another concept is illustrated by this same picture. We see that every point in the segment $\overline{BC}$ is also a point of $\overline{AD}$. We express this fact by saying that $\overline{BC}$ is a subset of $\overline{AD}$ or by saying that $\overline{BC}$ is contained in $\overline{AD}$. You should see that $\overline{AC}$ is a subset of $\overline{AD}$ and $\overline{BC}$ is a subset of $\overline{BD}$, but of the segments $\overline{AC}$ and $\overline{BD}$ neither is a subset of the other.

Other examples of subsets are:

> The set of boys in your school is a subset of the set of students in your school.
>
> The set $\{1, 5, 9\}$ is a subset of the set $\{1, 3, 5, 7, 9\}$.

In these examples the words "is a subset of" may be replaced by "is contained in."

Remember that in general for sets S and T, "S is a subset of T" or "S is contained in T" means that every member of S is also a member of T.

The concept of subset is extremely useful. One reason for this is that anything that is true of all members of a set is also true of all members of a subset. For example: if all the students in your school are American citizens then you may conclude that all the students in your class are American citizens. The reason that you can make this conclusion is that the set of students in your class is a subset of the set of students in your school.

Exercises 7-1b

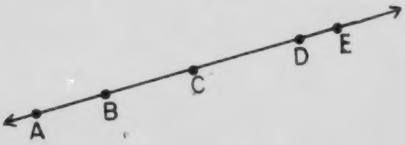


Figure 6

1. a. In Figure 6, is segment $\overline{BC}$ contained in segment $\overline{AD}$?
   b. Is $\overline{BC}$ a subset of $\overline{AD}$?
   c. Name other subsets of $\overline{AD}$ which are segments.

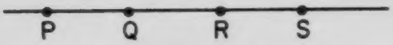
2. a. In Figure 6, is $\overrightarrow{CE}$ a subset of $\overleftrightarrow{CE}$?
   b. Name other rays which are subsets of $\overleftrightarrow{CE}$?

3. In each case, write the elements of the set that is the union of the given sets. Write the elements of the set that is the intersection of the given sets.
   a. R = {a, b, c, d, e}
      S = {a, e, i, o, u}
   b. P = {1, 3, 5, 7, 9, 11}
      Q = {1, 4, 9, 16}
   c. X = The whole numbers 1 through 10
      Y = The whole numbers 3 through 12
   d. A is the set of linemen on a football team.
      B is the set of backs on a football team.

4. Draw a line. Label three points of the line A, B, and C with B between A and C.
   a. What is $\overline{AB} \cap \overline{BC}$?
   b. What is $\overline{AC} \cap \overline{BC}$?
   c. What is $\overline{AB} \cup \overline{BC}$?
   d. What is $\overline{AB} \cup \overline{AC}$?
   e. What is $\overline{AC} \cup \overline{BC}$?
   f. What is $\overline{AB} \cup \overline{AB}$?

5. Let A and B be two points. Is it true that there is exactly one segment containing A and B? Draw a figure explaining this problem and your answer.

6. In the figure at the right



   a. What is $\overline{PR} \cup \overline{QR}$?
   b. What is $\overline{QR} \cup \overline{RS}$?
   c. What is $\overline{PS} \cup \overline{QS}$?
   d. What is $\overline{QR} \cup \overline{QS}$?
   e. What is $\overline{PR} \cup \overline{RS}$?
   f. What is $\overline{PR} \cup \overline{QS}$?
   g. What is $\overline{PQ} \cup \overline{QR}$?

7. Draw a vertical line ℓ. Label M and N two points to the left of ℓ. Label C a point to the right of ℓ. Is $\overline{MN} \cap \ell$ empty? Is $\overline{MC} \cap \ell$ empty? Explain.

8. a. Explain why for any sets A and B we have $A \cap B$ is contained in A while A is contained in $A \cup B$.

   b. Explain why: If A is contained in B and B is contained in C then A is contained in C.

9. Illustrate both parts of Problem 8:

   a. With sets of people.

   b. With sets of numbers.

   c. With geometrical sets.

10. A = {multiples of 10}

    B = {multiples of 2}

    C = {multiples of 5}

    a. How is B related to A?
    b. How is C related to A?
    c. How is $B \cap C$ related to A?

7-2. Separations.

We often find it convenient to separate sets of objects. For example, in a movie theatre we usually separate the seats on the left side from the seats on the right side by a center aisle. In baseball, we separate the points in fair territory from the points in foul territory by foul lines.

This idea of separation is also important in mathematics. Three important cases of this idea will be considered below:

In Figure 1, let plane ABC be the plane of a window in your classroom. This plane separates space into two sets:

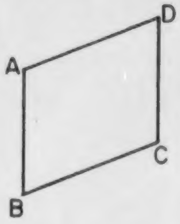


Figure 1

1. The set of points on your side of the plane of the window.
2. The set of points on the other side of the plane of the window.

The portions of space on the two sides of plane ABC are called half-spaces. The plane ABC itself, is not in either half-space.

Let R and S be any two points in space not in the plane ABC of the window. Then R and S are on the same side of the plane ABC if the intersection of $\overline{RS}$ and plane ABC is empty.

Also, R and S are on opposite sides of the plane ABC if the intersection of $\overline{RS}$ and plane ABC is not empty. In this case, there is a point of plane ABC on $\overline{RS}$.

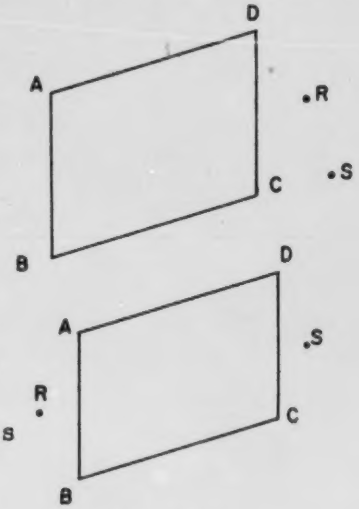


Figure 2

Thus, we have three types of separation:

Type 1-Any plane ABC separates space into two half-spaces.

If R and S are in the same half-space then $\overline{RS} \cap$ plane ABC is empty.

If R and S are in different half-spaces, $\overline{RS} \cap$ plane ABC is not empty.

We call plane ABC the boundary of each of the half-spaces.

Consider the plane XYZ. Do you see how the plane XYZ could be separated into two half-planes? What would be the boundary of the two half-planes?

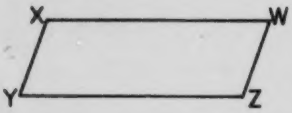


Figure 3

Do you agree that a line such as ℓ, separates XYZ into two half-planes. We call line ℓ the boundary of the two half-planes.

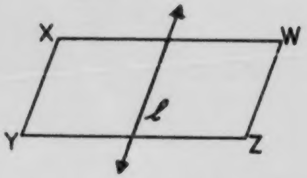


Figure 4

Look at Figure 5. It consists of the line ℓ and three points A, B, and C.

Is $\overline{AB} \cap \ell$ the empty set?
Is $\overline{BC} \cap \ell$ the empty set?
Is $\overline{AC} \cap \ell$ the empty set?

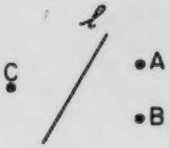


Figure 5

If your answers were yes, no, no, you were correct. Thus, we have

Type 2-Any line ℓ of plane XYZ separates the plane into two half-planes.

Recall that planes and lines extend without limit. Therefore each half-plane extends without limit.

We call the two half-planes into which a line separates a plane, the sides of the line. For example, in Figure 6, the line ℓ separates the plane ABC into two half-planes. The two half-planes are called the sides of ℓ. We name the sides of ℓ by saying the A-side of ℓ and the C-side of ℓ. In Figure 6, the B-side of ℓ = the A-side of ℓ.

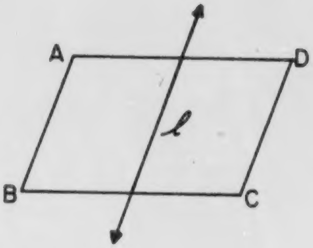


Figure 6

Now, consider the line ℓ . What is meant by a half-line? Do you agree that a point P separates line ℓ into two half-lines? We call point P the boundary of the half-lines.

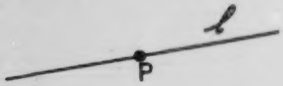


Figure 7

Type 3-*Any point P of line ℓ separates the line ℓ into two half-lines*.

In Figure 8, D separates ℓ so that A and C are on the same half-line and A and B are on different half-lines. Can you see that, if two points are on different half-lines of the same line, then the boundary must lie between them?

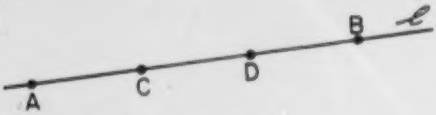


Figure 8

Exercises 7-2a

(Class Discussion)

1. Explain how each of the following may be used as an example of a separation:

   a. The plane of the net on a tennis court.
   b. First base on a baseball field.
   c. A movie screen.
   d. The right edge of your desk.
   e. A window in your living room.
   f. The 8-inch mark on a 12-inch ruler.
   g. The goal line on a football field.

2. In Figure 9, line ℓ and points P, Q, R and S are in one plane. Are the following statements true or false? Give reasons for your answer.
   a. The R-side of ℓ is the same as the S-side of ℓ.
   b. The S-side of ℓ is the same as the Q-side of ℓ.
   c. $\ell \cap \overline{PQ}$ is empty.
   d. $\ell \cap \overline{RS}$ is empty.
   e. $\ell \cap \overline{QS}$ is empty.
   f. $\ell \cap \overline{PR}$ is empty.

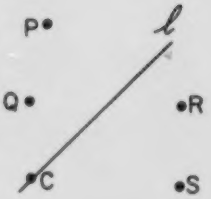


Figure 9

3. In Figure 10,
   a. Does $\overleftrightarrow{PQ}$ separate the plane ADC?
   b. Does $\overline{PQ}$ separate the plane ADC?
   Explain your answers.

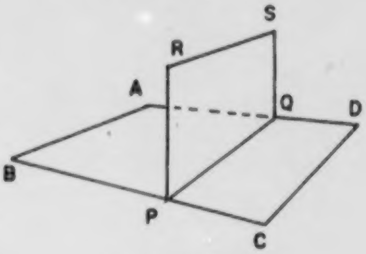


Figure 10

4. Draw two parallel horizontal lines k and m on your paper. Label point P on line m. Is every point on m on the P-side of k? Explain.

5. The idea of a plane separating space is similar to the idea of the surface of a box separating the inside from the outside. If P is a point on the inside and Q a point on the outside of a box, does $\overline{PQ}$ intersect the surface?

*6. a. Could the union of two half-lines be a line? Explain.
   b. Could the union of two half-planes be a plane? Explain.

*7. If A and B are points on the same side of plane RST (in space) must $\overleftrightarrow{AB} \cap$ plane RST be empty? Can $\overleftrightarrow{AB} \cap$ plane RST be empty?

A ray is a half-line together with its endpoint. A ray has one endpoint. Thus, a ray without its endpoint is a half-line. We usually draw a ray like this, →. If A is the endpoint of a ray and B is another point of the ray we denote the ray by $\overrightarrow{AB}$. Note that $\overrightarrow{BA}$ is not the same as $\overrightarrow{AB}$. We use the term ray in the same sense in which it is used in "ray of light."

In everyday language, we sometimes do not use the words lines, rays, and segments as words with different meanings. In geometry, we should be careful to use them precisely. A "line of sight" is really a ray because it has an endpoint. Thus, you do not describe somebody as in your line of sight if he is behind you.

The right field foul line in baseball refers to the union of a segment and a ray. The segment extends from home plate through first base to the bottom of the ball park fence. It stops at the fence. The ray starts on the ground and goes up the fence.

Exercises 7-2b

1. Use Figure 11 to explain the meaning of:
    a. $\overleftrightarrow{RS}$
    b. $\overline{RS}$
    c. $\overrightarrow{RS}$

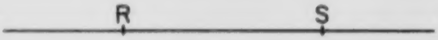


Figure 11

2. In Figure 12 :
    a. What is $\overline{PL} \cup \overline{LK}$?
    b. What is $\overline{PL} \cap \overline{LK}$?
    c. What is $\overrightarrow{LP} \cup \overrightarrow{LK}$?

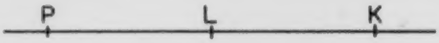


Figure 12

3. Draw a line containing points A, B, and C in order from left to right.
   a. What is $\overrightarrow{AB} \cap \overrightarrow{BA}$?
   b. What is $\overrightarrow{BA} \cap \overrightarrow{CB}$?
   c. What is $\overrightarrow{AC} \cap \overrightarrow{BA}$?
   d. What is $\overrightarrow{BA} \cup \overrightarrow{BC}$?
   e. What is $\overrightarrow{BA} \cup \overrightarrow{CB}$?

4. Draw a horizontal line. Label four points on it A, B, C, and D in that order from left to right. Name two rays (using pairs of these points to name them):
   a. whose union is the line.
   b. whose union is not the line but contains A, B, C, and D.
   c. whose union does not contain A.
   d. whose intersection is a point.
   e. whose intersection is empty.

7-3. Angles and Triangles.

Angles.

An *angle* is a set of points consisting of two rays with a common endpoint and not both in the same straight line. For example, in Figure 1, $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are two rays. B is the common endpoint and A, B, and C are not on the same line. Thus, the set of points consisting of all the points of $\overrightarrow{BA}$ together with all the points of $\overrightarrow{BC}$ is called the angle ABC.

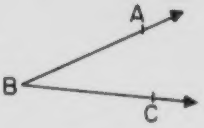


Figure 1

An angle is the union of two rays with a common endpoint, and not both in the same straight line. The common point B is called the *vertex* of the angle. The rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are called the rays (or sometimes the sides) of the angle. An angle has exactly one vertex and exactly two rays.

Consider the angle drawn in Figure 2. Remember that R represents a point and that $\overrightarrow{RS}$ and $\overrightarrow{RT}$ represent rays. Similarly, Figure 2 represents an angle.

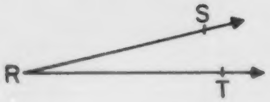


Figure 2

It is often necessary to name angles. The angle in Figure 2 is named $\angle$ SRT or $\angle$ TRS. The letter at the vertex is always written between the other two letters.

Exercises 7-3a

(Class Discussion)

1. Draw an angle ABC.

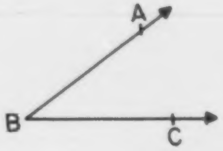


Figure 3

2. You can see that the angle separates the plane.

3. Shade the portion of the plane that appears outside the angle, as shown. This shaded portion is called the exterior of the angle.

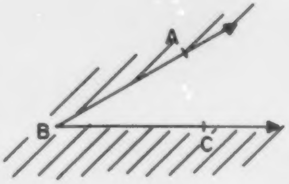


Figure 4

4. Shade the portion of the plane that appears inside the angle, as shown. This shaded portion is called the interior of the angle.

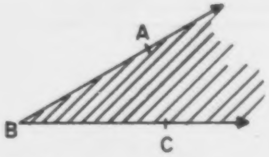
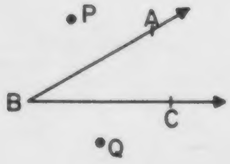


Figure 5

5. Explain the following statement.
"The interior of $\angle$ ABC can be defined as the intersection of the A-side of $\overleftrightarrow{BC}$ and the C-side of $\overleftrightarrow{AB}$."

Note that the interior of an angle is the intersection of two half-planes. The exterior of an angle can never be the intersection of two half-planes. The interior of an angle does not contain the angle. The exterior of an angle is the set of all points of the plane not on the angle nor in the interior.

6. Another way to indicate the exterior of $\angle$ ABC is to say that it is the union of the P-side of $\overleftrightarrow{AB}$ and the Q-side of $\overleftrightarrow{BC}$.



Exercises 7-3b

1. In Figure 6,
   a. Name the angle in two ways.
   b. Name the vertex of the angle.
   c. Name the rays (or sides) of the angle.

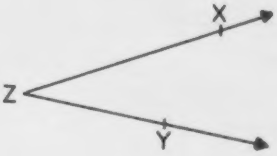


Figure 6

2. Label three points A, B, and C not all on the same line. Draw $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$.
   a. Shade the C-side of $\overleftrightarrow{AB}$.
   b. Shade the A-side of $\overleftrightarrow{BC}$.
   c. What set is now doubly shaded?

3. In Figure 7, what is:
   a. $\overrightarrow{VT} \cup \overrightarrow{VW}$?
   b. $\overleftrightarrow{TV} \cap \overleftrightarrow{WV}$?
   c. $\overrightarrow{VT} \cap \overrightarrow{VW}$?

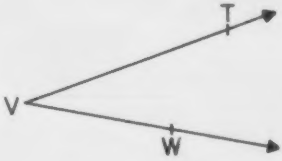


Figure 7

Triangles

Let A, B, and C be three points not all on the same straight line. The triangle ABC, written as Δ ABC, is the union of $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$. You will recall that the union of two sets consists of all the elements of one set together with all the elements of the other set. In this case, we have the union of three sets. The union of $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$ consists of all the elements of the three sets taken together.

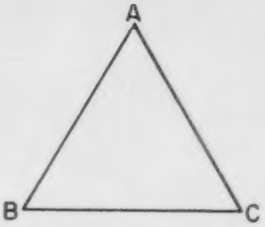


Figure 8

The triangle ABC is the set of points consisting of A, B, and C, and all points of $\overleftrightarrow{AB}$ between A and B, all points of $\overleftrightarrow{AC}$ between A and C, and all points of $\overleftrightarrow{BC}$ between B and C. The points A, B, and C are the vertices of Δ ABC. We say "vertices" when we talk about more than one vertex.

Angles of a Triangle

Do you see that the sides of a triangle are contained in the triangle? This is true because the sides of a triangle are segments.

Are the angles of a triangle contained in the triangle? No! In the figure $\angle$ RST is the union of $\overrightarrow{SR}$ and $\overrightarrow{ST}$. As you can see the angle extends beyond the sides of the triangle.

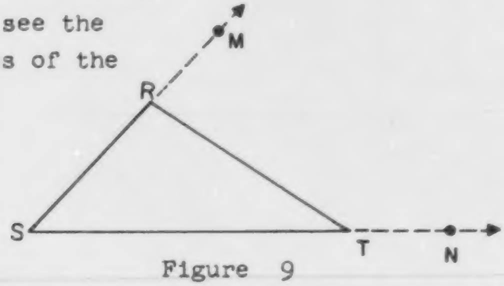


Figure 9

Note that a triangle is a set of points in exactly one plane.

Every point of the triangle RST is in the plane RST.

Would you say that Δ RST separates the plane in which it lies? It does.

The Δ RST has an interior and an exterior. The interior is the intersection of the interiors of the three angles of the triangle. The exterior is the set of all points of plane RST not on Δ RST or in the interior of Δ RST.

Exercises 7-3c

1. Label three points A, B, and C not all on the same line. Draw $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, and $\overleftrightarrow{BC}$.
   a. Shade the C-side of $\overleftrightarrow{AB}$. Shade the A-side of $\overleftrightarrow{BC}$. What set is now doubly shaded?
   b. Shade the B-side of $\overleftrightarrow{AC}$. What set is now triply shaded?

2. Label three points X, Y, and Z not all on the same line.
   a. Draw $\angle$ XYZ and $\angle$ XZY. Are they different angles? Why?
   b. Draw $\angle$ YXZ. Is this angle different from both of the other two you have drawn?
   c. Each angle is a set of points in exactly one plane. Why is this true?

3. Draw a triangle ABC.
   a. In the triangle, what is $\overline{AB} \cap \overline{AC}$?
   b. Does the triangle contain any rays or half-lines? Why?
   c. In the drawing extend $\overline{AB}$ in both directions to obtain $\overleftrightarrow{AB}$. What is $\overline{AB} \cap \overleftrightarrow{AB}$?
   d. What is $\overleftrightarrow{AB} \cap \Delta ABC$?
   e. What is $\overrightarrow{AB} \cup \overrightarrow{AC}$?

4. In Figure 10
   a. What is $\overline{YW} \cap \Delta ABC$?
   b. Name the four triangles in the figure.
   c. Which of the labeled points, if any, are in the interior of any of the triangles?

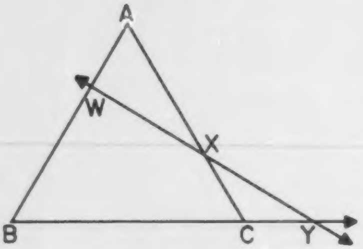


Figure 10

d. Which of the labeled points, if any, are in the exterior of any of the triangles?

e. Name a point on the same side of $\overleftrightarrow{WY}$ as C and one on the opposite side.

5. On your paper, make a copy of Figure 10.

a. Label a point P not in the interior of any of the triangles.

b. Label a point Q inside two of the triangles.

c. If possible, label a point R in the interior of $\triangle ABC$, but not in the interior of any other of the triangles.

6. If possible, make sketches in which the intersection of a line and a triangle is:

a. The empty set

b. A set of one element

c. A set of two elements

d. A set of exactly three elements

e. A line segment.

7. If possible, make sketches in which the intersection of two triangles is:

a. The empty set

b. Exactly one point

c. Exactly two points

d. Exactly three points

e. Exactly four points

f. Exactly five points

g. A line segment

h. Two line segments

i. Three line segments

8. In Figure 11, what are the following:

a. $\angle ABC \cap \overleftrightarrow{AC}$

b. $\triangle ABC \cap \overleftrightarrow{AB}$

c. $\ell_1 \cap \angle ACB$

d. $\overline{AB} \cap \ell_2$

e. $\overline{BC} \cap \triangle ABC$

f. $\overrightarrow{BC} \cap \angle ACB$

g. $\angle ABC \cap \triangle ABC$

h. $\overrightarrow{AC} \cup \overrightarrow{CB}$

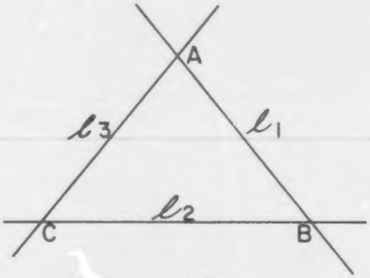


Figure 11

*9. In a plane if two triangles have a common side must their interiors intersect? If three triangles have a common side, must some two of their interiors intersect? Make drawings to explain your answers.

10. BRAINBUSTER: Draw $\angle ABC$. Label points X and Y in the interior and P and Q in the exterior.
   a. Must every point of $\overline{XY}$ be in the interior?
   b. Is every point of $\overline{PQ}$ in the exterior?
   c. Can you find points R and S in the exterior so that $\overline{RS} \cap \angle ABC$ is empty?
   d. Can $\overline{XP} \cap \angle ABC$ be empty?

---

*7-4. One-to-One Correspondence. (Optional)

In the first part of Chapter 3 we studied about one-to-one correspondence. We learned that early man kept a record of the number of sheep in his flock as follows: For each sheep he put a stone in a pile. In order to make sure that no sheep were missing he took a stone out of the pile as each sheep went into the pen. If there were no stones left in the pile when the last sheep was in the pen he knew that no sheep were missing. This process of *matching* each member of one set with a member of another set is an example of one-to-one correspondence.

Let us consider other examples of one-to-one correspondence, or matching. When Mrs. Barber served tea she put a set of saucers on the table, one for each guest. Susan Barber then placed a cup on each saucer. Thus, there was a one-to-one correspondence between the set of saucers and the set of cups on the table.

At the start of a major league baseball game nine players ran onto the field and took their positions. There was a one-to-one correspondence between the set of players and the set of positions.

Let U be the set of human noses in your classroom.
Let V be the set of humans in your classroom.

Is there a one-to-one correspondence between U and V? For each nose is there a matching human? For each human is there a matching nose?

In showing that two sets X and Y are in one-to-one correspondence we must show that:

a. For each element in X there is a corresponding element in Y.

and

b. Each element in Y corresponds to a given element in X.

Let us see how a one-to-one correspondence is used in geometry.

Follow the directions. Answer the questions as you go along.

Exercises 7-4a

(Class Discussion)

1. Draw a line and label it ℓ .
2. Locate a point above line ℓ and label it P.
3. Locate a point on line ℓ and label it A.
4. Draw $\overleftrightarrow{PA}$.
   a. Does $\overleftrightarrow{PA}$ intersect line ℓ ?
   b. How many elements are there in the intersection set of $\overleftrightarrow{PA}$ and ℓ ?
5. Locate a second point on line ℓ and label it B.
6. Draw $\overleftrightarrow{PB}$.
   a. Does $\overleftrightarrow{PB}$ intersect ℓ ?
   b. How many elements are in the intersection set of $\overleftrightarrow{PB}$ and ℓ ?
7. Locate a third point C on line ℓ .
8. Draw $\overleftrightarrow{PC}$.
   a. Does $\overleftrightarrow{PC}$ intersect ℓ ?
   b. How many elements are in the intersection set of $\overleftrightarrow{PC}$ and ℓ ?

9. a. Is it possible to locate more points on ℓ ?

   b. Through each additional point marked on ℓ can you draw a line that also passes through point P?

10. Let all lines which intersect ℓ and pass through P be elements of a set called K.

   a. How many elements of K have been drawn so far?

   b. How many points have been located on line ℓ so far?

   c. Is there a point on ℓ for each element of K drawn so far?

   d. Is there an element of K for each point located on ℓ ?

   e. Can you match "one-to-one" the elements of K and the points located so far?

   f. Do you think that if more points were located on ℓ and more elements of K were drawn that a one-to-one correspondence between the sets could be shown?

   g. Supply the missing words:
   To a line through P and intersecting ℓ there corresponds a ________ on ℓ, and to a ________ on ℓ there corresponds a __________ through P and intersecting ℓ .

Have you found that there is a one-to-one correspondence between the set K of lines and the set ℓ of points?

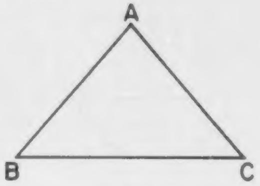
Exercises 7-4b

1. Each desk in a classroom is assigned to a pupil.

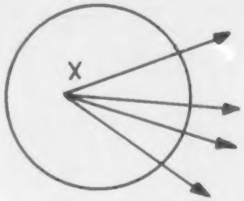
   a. When all pupils are present, is there a one-to-one correspondence between the pupils and desks in the classroom?

   b. Is there always a one-to-one correspondence between the pupils and desks in a classroom? Explain your answer.

2. If you have a full set of fingers (including thumbs):
   a. Is there a one-to-one correspondence between the fingers on your left hand and the fingers on your right hand?
   b. Copy and complete: For each finger on my left hand there is a _________ finger on my _________ _________.
3. Two basketball teams are playing on a court. Show that there is a one-to-one correspondence between the members of one team and the members of the other team.
4. In the triangle ABC show that there is a one-to-one correspondence between points A, B, and C and the sides opposite these points.



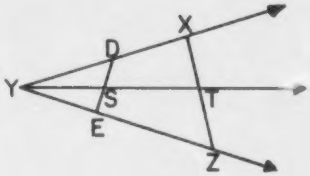
5. Draw a circle as shown at the right. Label a point X in the interior of the circle. Let R be the set of all rays having X as an endpoint. We understand that the elements of R are in the same plane as the circle. Draw several rays of set R. Copy and complete the following:



   a. For every point of the circle there is _________ ray of set R cutting the circle.
   b. For every ray of set R, there is _________ _________ in which the ray cuts the circle.
   c. Is there a one-to-one correspondence between set R and the circle?

6. In the diagram below, $\angle XYZ$ is an angle, $\overrightarrow{YX}$, $\overrightarrow{YT}$, and $\overrightarrow{YZ}$ are elements of set K. Set K is the set of all the rays through Y which do not contain points in the exterior of $\angle XYZ$.

$\overline{DE}$ and $\overline{XZ}$ are segments joining points on $\overline{YZ}$ with points on $\overline{YX}$.



a. Show that there is a one-to-one correspondence between K and $\overline{DE}$.

b. Show that there is a one-to-one correspondence between K and $\overline{XZ}$.

c. Show that there is a one-to-one correspondence between $\overline{DE}$ and $\overline{XZ}$.

*7. Show that there is a one-to-one correspondence between the set of even whole numbers and the set of odd whole numbers.

*8. Show that there is a one-to-one correspondence between the set of even whole numbers and the set of whole numbers.

7-5. Simple Closed Curves.

In newspapers and magazines you often see graphs like those in Figures 1 and 2.

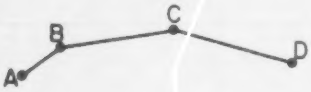


Figure 1

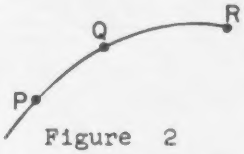


Figure 2

These graphs represent curves. We shall consider curves to be special types of sets of points. Sometimes, paths that wander around in space are thought of as curves. But in this section we shall consider only curves that are in one plane. Such curves may be drawn on a chalkboard or on a sheet of paper.

A *curve* is a set of points which can be represented by a pencil drawing without lifting the pencil off the paper. Segments and triangles are examples of curves we have already studied. Curves may or may not contain portions that are straight.

One important type of curve is called a *broken-line curve*. It is the union of several line segments. That is, it consists of all the points on several line segments. Figure 1 represents a broken-line curve. A, B, C and D are marked as points on the curve. We also say that the curve *contains* or *passes* through these points. Figure. 2 also represents a curve. In Figure 2 points P, Q and R are marked on the curve. Of course, we think of the curve as containing many points other than P, Q, and R.

A curve is said to be a *simple closed curve* if:

a. the drawing starts and stops at the same point.
b. no other point is touched twice by the pencil mark.

Figures 3, 4, 5, and 6 are examples of simple closed curves.

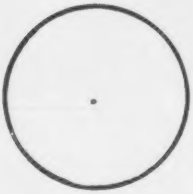


Figure 3

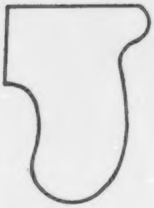


Figure 4

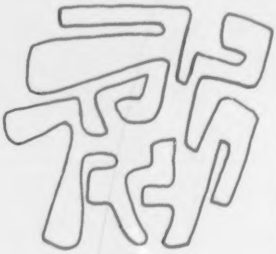


Figure 5

Figure 6

Figures 3a, 4a, 5a, and 6a are examples of curves which are not simple closed curves.

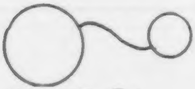


Figure 3a

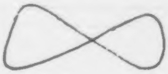


Figure 4a

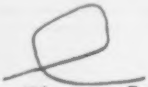


Figure 5a

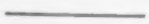


Figure 6a

Figure 7 represents two simple closed curves.

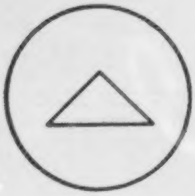


Figure 7

The boundary of a state like Iowa or Utah on an ordinary map represents a simple closed curve. A fence which extends all the way around a cornfield or a city park suggests a simple closed curve.

You can see that a simple closed curve has an interior and an exterior. Also, any line or curve containing a point in the interior and a point in the exterior must intersect the closed curve.

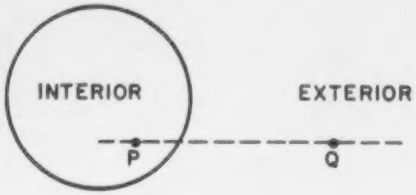


Figure 8

In Figure 8, P is in the interior of the closed curve and B is in the exterior of the closed curve. The dotted line which contains P and Q must cut the curve.

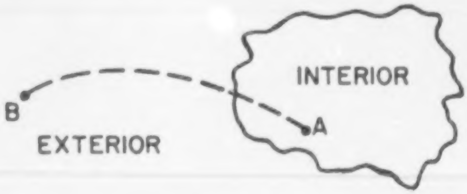


Figure 9

In Figure 9, A is in the interior of the closed curve and B is in the exterior of the closed curve. The dotted curve which contains A and B must cut the curve.

Any two points in the interior or any two points in the exterior of a simple closed curve may be joined by a broken-line curve which does not intersect the simple closed curve.

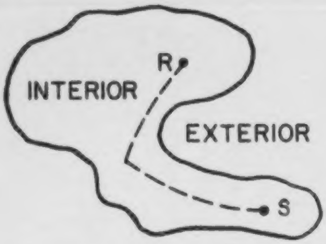


Figure 10

In Figure 10, R and S are points in the interior of a simple closed curve. R and S are joined by a broken-line curve (dotted line) which does not intersect the simple closed curve.

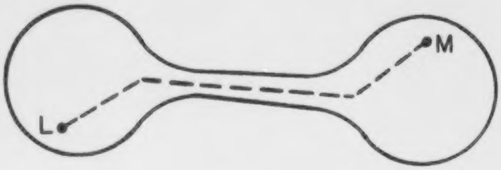


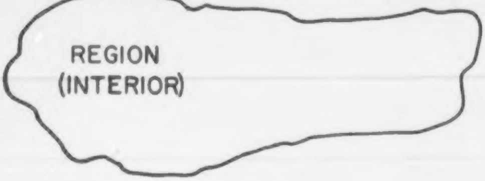
Figure 11

In Figure 11, L and M are points in the interior of a simple closed curve. L and M are joined by a broken-line curve (dotted line) which does not intersect the simple closed curve.

We call the interior of a simple closed curve a *region*.

We call the interior of a simple closed curve together with its boundary a *closed region*.

Closed Region (Interior and Boundary)

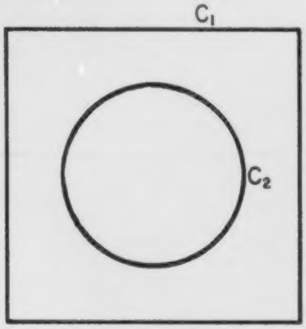


Exercises 7-5

1. Draw the simple closed curves described below:
   a. one which is the union of four segments.
   b. one which is the union of five segments.
   c. one which is the union of three segments.

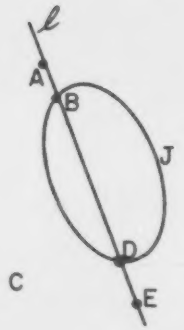
2. Draw a figure representing two simple closed curves whose intersection is exactly two points.

3. In the figure at the right, the two simple closed curves are represented as C_1 and C_2.
   a. Make a copy of this figure.
   b. Shade the exterior of C_2 with horizontal lines.
   c. Shade the interior of C_1 with vertical lines.
   d. Using the word "intersection" describe in words the portion of the plane that is doubly shaded.
   e. Describe the portion of the plane that is singly shaded.



4. Look at a map of the United States.
   a. Does the boundary of Colorado represent a simple closed curve?
   b. Does the boundary of Arizona represent a simple closed curve?
   c. Does the union of the boundaries of Colorado and Arizona represent a simple closed curve? Why?

5. The line ℓ and the simple closed curve J are shown in the figure at the right.
   a. What is $J \cap \ell$?
   b. Draw a similar figure and shade the intersection of the interior of J and the C-side of ℓ.
   c. Describe in terms of rays the set of points on ℓ *not* in the interior of J.



6. BRAINBUSTER. Draw two simple closed curves whose interiors intersect in three different regions.

7. BRAINBUSTER. Think of X and Y as bugs which can crawl anywhere on a floor. (a) Draw a picture showing a curve which could serve as a fence to separate X from Y. (b) Draw another picture in which it is hard to tell whether the curve separates X from Y.

7-6. *Summary*.

Many of the practical uses of geometry are based upon the angle and the triangle. In order to work with the angle and the triangle we will need to understand the meaning of segment, ray, and union of sets.

A *segment* is a part of a line together with its endpoints.

A *ray* is a half-line together with its endpoint.

If every member of a certain set is also a member of (or contained in) a second set, the first set is called a *subset* of the second.

The *union* of two sets is a set consisting of all the elements of the first set together with all the elements of the second set.

Using these ideas we can now explain the meaning of angle and triangle.

An angle is a set of points consisting of two rays with an endpoint in common and both rays not on the same straight line.

The triangle ABC, written Δ ABC, is the union of $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$ provided that A, B, and C are not on the same straight line.

A half-line is a ray without its endpoint.

A region is a set of points in the interior of a simple closed curve.

The following ideas are useful in mathematics:

Separation

In this chapter, the following three kinds of separation are considered:

1. Any plane separates space into two half-spaces.
2. Any line of a plane separates the plane into two half-planes.
3. Any point of a line separates the line into two half-lines.

One-To-One Correspondence

Two sets can be shown to be in one-to-one correspondence if:

a. Each element of the first set can be matched with an element of the second set, and

b. Each element of the second set is matched with a given element of the first set.

Simple Closed Curve

A curve is said to be a simple closed curve if it can be drawn:

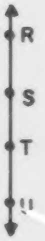
a. So that it starts and stops at the same point.

b. No other point is touched twice by the pencil mark.

7-7. Chapter Review.

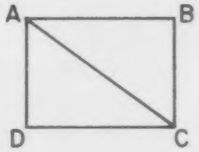
Exercises 7-7

1. Draw a vertical line. Label four points on it R, S, T, U (as shown). Name two rays:
   a. Whose endpoints are S.
   b. Whose union is the line.
   c. Whose union is not the line but contains R, S, T, and U.
   d. Whose union does not contain R.
   e. Whose intersection is a point.
   f. Whose intersection is empty.

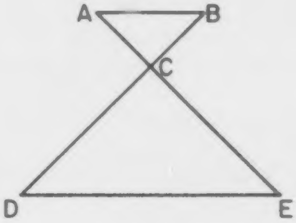


2. Name ten subsets of $\overleftrightarrow{RU}$.

3. In the drawing at the right
   a. Explain the difference between $\overleftrightarrow{AD}$ and $\overline{AD}$.
   b. Explain the difference between $\overline{AB}$ and $\overrightarrow{AB}$.
   c. What is the intersection of $\overleftrightarrow{AD}$ and $\overleftrightarrow{CD}$?



4. In the figure at the right
   a. What is $\overline{AB} \cap \overline{AC}$?
   b. What is $\overline{BC} \cup \overline{CD}$?
   c. What is $\overleftrightarrow{AE} \cap \overleftrightarrow{BD}$?
   d. What is $\overline{AB} \cap \overline{DE}$?



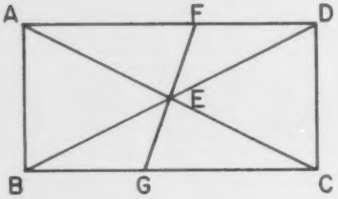
5. Draw a horizontal line. Label four points on it P, Q, R, and S in that order from left to right. Name two segments
   - a. Whose intersection is a segment.
   - b. Whose intersection is a point.
   - c. Whose intersection is empty.
   - d. Whose union is not a segment.

6. If $R = \{1, 3, 5, 7, 9, 10\}$

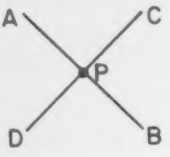
   And $S = \{4, 7, 10\}$

   Write
   - a. $R \cap S$
   - b. $R \cup S$

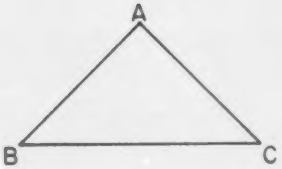
7. In the figure at the right
   - a. Name a point on the B-side of $\overleftrightarrow{AC}$.
   - b. Name a point on the F-side of $\overleftrightarrow{BD}$.
   - c. What is $\overleftrightarrow{AC} \cap \overline{BD}$?
   - d. What is $\overline{AB} \cap \overline{DC}$?
   - e. What is $\overline{EF} \cup \overline{EG}$?
   - f. What is $\overrightarrow{EB} \cup \overrightarrow{EG}$?



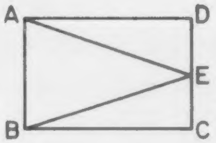
8. Explain how each of the following may be used as an example of a separation.
   - a. The plane of a shelf in a bookcase.
   - b. The left sideline on a basketball court.
   - c. The plane of a store window.
   - d. Point P.



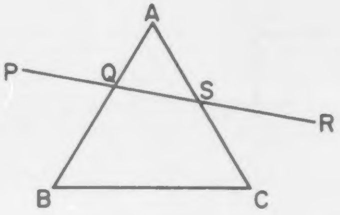
9. Draw $\triangle ABC$.
   a. Name the angle in the triangle whose vertex is B.
   b. What is $\overline{AB} \cap \overline{AC}$?
   c. Are the sides of the triangle rays, half-lines, or segments?
   d. In the drawing, extend $\overline{AB}$ in both directions to obtain $\overleftrightarrow{AB}$. What is $\overline{AB} \cap \overleftrightarrow{AB}$?
   e. What is $\overline{AB} \cap \triangle ABC$?



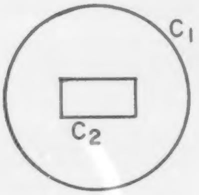
10. In the figure at the right, what is
   a. $\overleftrightarrow{AE} \cap \overleftrightarrow{EB}$?
   b. $\overline{DE} \cup \overline{EC}$?
   c. $\triangle EBC \cap \overleftrightarrow{DE}$?
   d. $\overrightarrow{EA} \cup \overrightarrow{ED}$?
   e. $\overline{AD} \cap \overline{BC}$?



11. In the diagram at the right.
   a. Point Q separates _____.
   b. C is on the _____ side of $\overleftrightarrow{PR}$.
   c. $\overrightarrow{QB} \cup \overrightarrow{QS}$ is _____.
   d. $\triangle ABC \cap \overleftrightarrow{PR}$ is _____.
   e. $\angle CBA \cap \overline{PR}$ is _____.

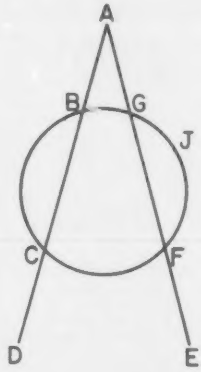


12. In the figure at the right, the two simple closed curves are represented by C_1 and C_2.
   a. Make a copy of this figure.
   b. Shade the exterior of C_2 with vertical lines.
   c. Shade the interior of C_1 with horizontal lines.
   d. Using the symbol for intersection indicate the region that is doubly shaded.

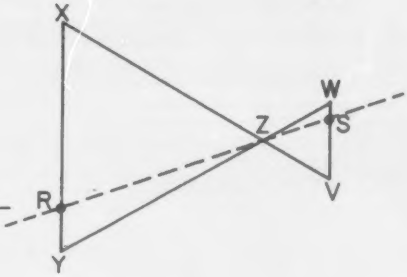


13. S is the set of states in the United States. C is the set of state capitals in the United States. Describe a one-to-one correspondence between set C and set S.

14. In the figure at the right, J is a simple closed curve. What is
    a. $\overleftrightarrow{DA} \cap \overleftrightarrow{FA}$?
    b. $J \cap \overrightarrow{AE}$?
    c. A point outside J?
    d. $\overrightarrow{AC} \cup \overrightarrow{AF}$?
    e. $\overline{AB} \cup \overline{BC}$?



15. By drawing lines through point Z (like $\overleftrightarrow{RS}$) show how to set up a one-to-one correspondence between the set of points on $\overline{XY}$ and the set of points on $\overline{WV}$.



7-8. Cumulative Review.

Exercises 7-8

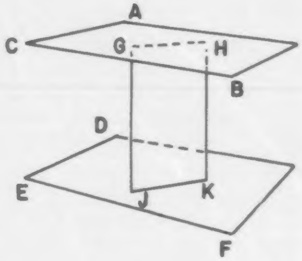
Tell if statements 1-8 are True or False.

1. The numeral for "seven" would look the same written in base seven or base ten.
2. 0 ÷ 0 is meaningless.
3. The numeral following 33_{four} is 100_{four}.
4. Some odd numbers are divisible by two.
5. The difference between any two prime numbers greater than 5 is always an even number.
6. A set is closed under addition if the sum of any two elements is an element of the set.
7. A common denominator of two fractions does not have to be a multiple of the denominators of both of the given fractions.

8. If a line contains two different points of a plane, it lies in the plane.

9. 3140_{eight} is how many times as large as 314_{eight}?

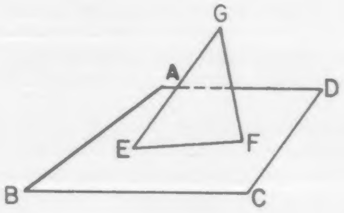
10. Find a whole number which can be used for x to make the following true:
    a. $7 + x = 15$
    b. $1 + x = x + 1$
    c. $x = 0 \div 3$



11. In the figure at the right
    a. What is $\overleftrightarrow{GJ} \cap$ plane DEF?
    b. What is $\overleftrightarrow{GH} \cap$ plane DEF?
    c. What is plane ACB $\cap$ plane DEF?
    d. What is $\overleftrightarrow{JK} \cap$ plane DEF?
    e. What is $\overleftrightarrow{HK} \cap$ plane ACB?

12. Which of these: 2, 3, 4, 5, 6, 9, 10 is a factor of
    a. 60
    b. 165

13. Perform the indicated operations and simplify
    a. $\frac{2}{7} + \frac{3}{7}$
    b. $\frac{5}{7} + \frac{2}{3}$
    c. $\frac{9}{10} - \frac{5}{6}$
    d. $\frac{3}{8} \cdot \frac{4}{7}$



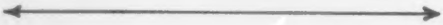
14. In the diagram on the right
    a. What is plane ABC $\cap$ $\overline{EF}$?
    b. What is $\overleftrightarrow{EG} \cap$ plane ADC?
    c. Explain why $\overleftrightarrow{GF}$ and $\overleftrightarrow{GE}$ are in the same plane.
    d. Explain why $\overleftrightarrow{EF}$ is in the same plane as $\overleftrightarrow{GE}$ and $\overleftrightarrow{GF}$.

Chapter 8

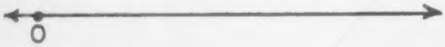
RATIONAL NUMBERS AND THE NUMBER LINE

8-1. *The Number Line*.

Do you remember how the number line is formed? You start with a line.

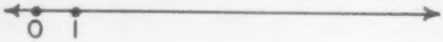


Then you select a point on the line which is called "0".

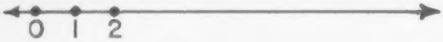


Next you choose a distance which is called a *unit of distance*.

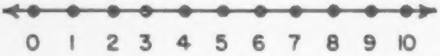
Using this unit of distance and starting at 0 you mark off a point to the right of 0 which is called "1".



Starting with 1 you measure off your unit of distance to find a point which is called "2".



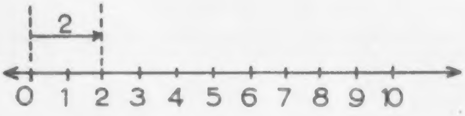
And so on:



The counting number at any point shows how many segments of unit distance are measured off from 0 to the point. Hence "4" indicates the point 4 units of distance from 0.

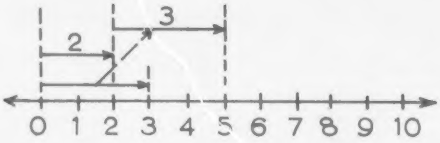
The number line may be used to "picture" the operations on the counting numbers. Suppose you want to add 2 + 3.

Start with 0 on the number line and mark off 2 units of distance. This is indicated by an arrow. The head of the arrow falls on point 2.

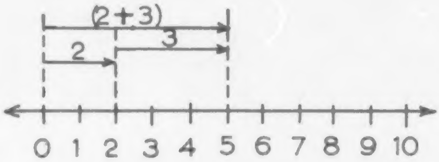


(In order to have a diagram which is easy to read place the arrows above the number line.)

Next mark off an arrow representing 3 and transfer this arrow to a position beginning at the head of arrow 2 and going to the right.

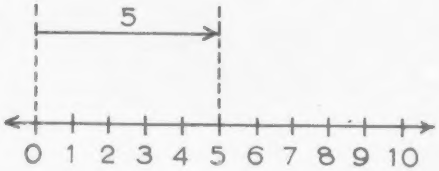


The sum (2 + 3) is represented by an arrow which begins at 0 and ends at the head of the "arrow 3". The head of this arrow is at 5, the sum of 2 + 3.

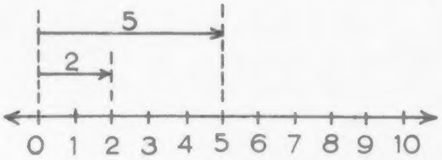


You can also picture a subtraction problem on the number line. Remember that the two problems 2 + 3 = 5 and 5 - 2 = 3 are just two ways of saying the same thing. For this reason the subtraction problem 5 - 2 = 3 should have the same diagram as 2 + 3 = 5.

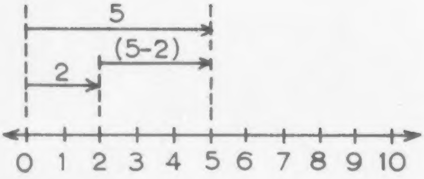
First represent 5 by an arrow starting at 0.



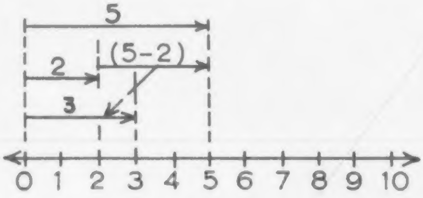
Then represent 2 by an arrow starting at 0.



The difference (5 - 2) is represented by an arrow starting at the head of "arrow 2" and ending at the head of "arrow 5".



To read this number (5 - 2) transfer the arrow back to 0. The arrow corresponds to 3. Since the length of this arrow is 3 units you can do this step mentally.)



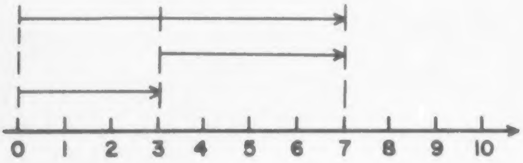
There are other ways that subtraction can be pictured on the number line but this method will be very useful later on. Here (5 - 2) is the number which, if added to 2, gives 5.

Notice that all the arrows in your diagrams point to the right and all the numbers which they represent correspond to points to the right of 0 on the number line.

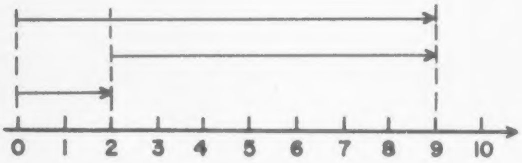
Exercises 8-1a
(Class Discussion)

State the addition problem and the corresponding subtraction problem which each diagram represents.

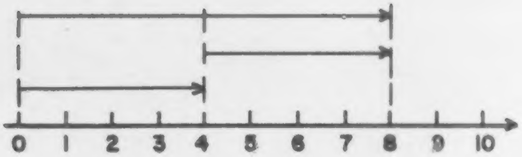
1.



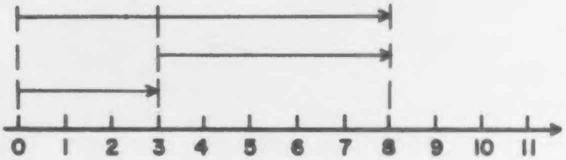
2.



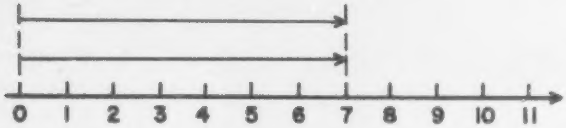
3.


0
1
2
3
4
5
6
7
8
9
10

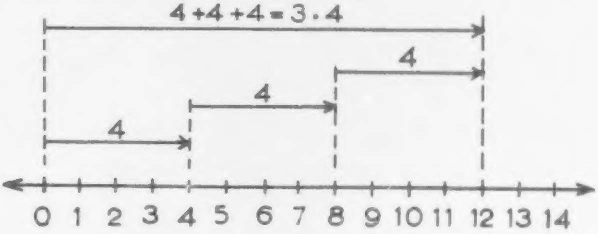
4.


0
1
2
3
4
5
6
7
8
9
10
11

*5.


0
1
2
3
4
5
6
7
8
9
10
11

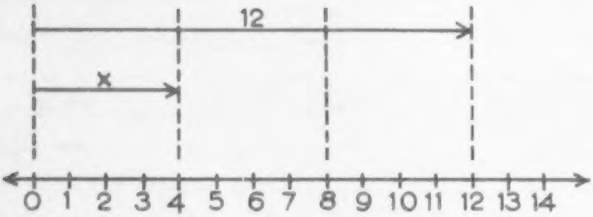
The number line can be used to multiply and divide. To show the problem $3 \cdot 4$, start at 0 and mark off three segments of 4 units each. The product is represented by the arrow from 0 to the head of the third arrow.


4+4+4=3·4
4
4
4
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14

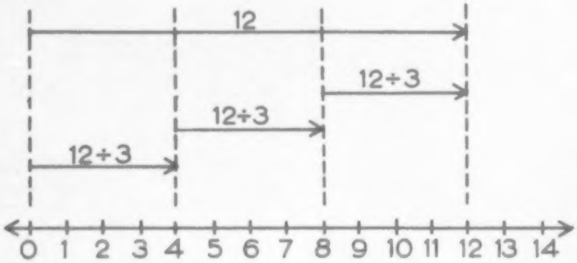
The corresponding division problem is

$$12 \div 3 = 4.$$

This is the same as the previous multiplication problem, since $12 \div 3 = 4$ means $12 = 3 \cdot 4$. The diagrams for these two problems are the same. Here we start at 0 and draw the arrow of 12 units.


12
x
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14

You want to divide this segment into three equal parts. What is the length of each of these three equal parts? It is the number $12 \div 3$. This is indicated on the diagram.


12
12÷3
12÷3
12÷3
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14

Exercises 8-1b

In each of the following problems draw a copy of the number line and show how to indicate the given operation.

1. a. Show how to find $4 + 3$

   b. Show how to find $7 - 4$

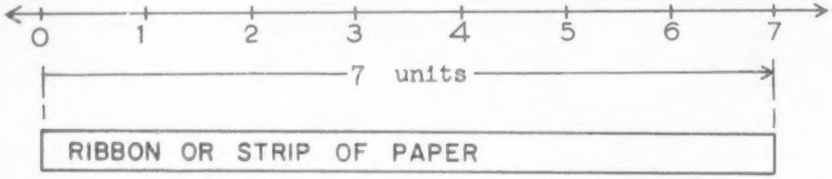
2. a. Show how to find $5 \cdot 2$

   b. Show how to find $10 \div 5$

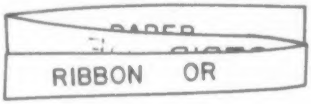
8-2. Locating Rational Numbers on the Number Line.

In the preceding section you represented the problem of dividing 12 by 3 on the number line. You divided a segment 12 units long into 3 segments of equal length. Can you divide 7 by 3? This presents some difficulties. The reason is that 7 ÷ 3 is not a whole number and the only numbers marked on our number line are the whole numbers. You can proceed in the same way as you did in dividing 12 by 3.

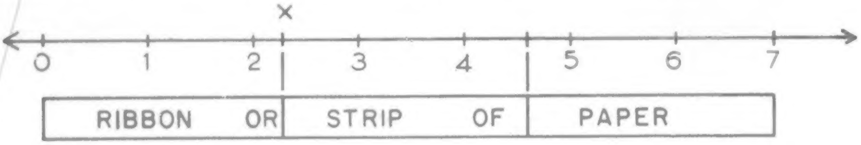
Take a piece of ribbon (or a strip of paper) 7 units in length.


0
1
2
3
4
5
6
7
7 units
RIBBON OR STRIP OF PAPER

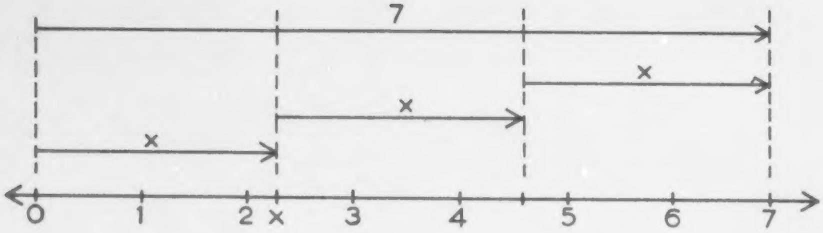
Fold the ribbon like this


PAPER
RIBBON OR

into three pieces of equal length and crease it at the folds. Unfold the ribbon and lay it between 0 and 7 on the number line. Transfer the creases onto the number line.


x
0
1
2
3
4
5
6
7
RIBBON OR
STRIP OF
PAPER

The segment from 0 to 7 has been divided into three segments of equal length. Label as "x" the point where the first crease fell.



The letter x then denotes the length of the segment from 0 to the point labeled "x". Each of the three segments has length x and therefore

$$3 \cdot x = 7.$$

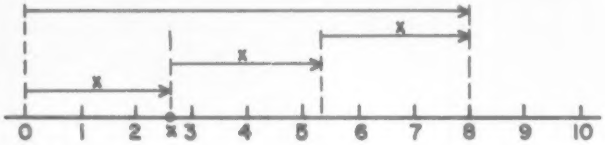
Other ways of saying this are

$$x = 7 \div 3 \qquad \text{or} \qquad x = \frac{7}{3}.$$

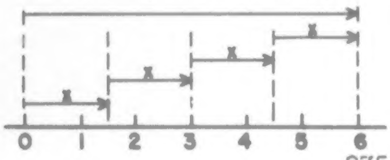
The point marked x is called $\frac{7}{3}$. In the same way you can represent other rational numbers on the number line.

Exercises 8-2
(Class Discussion)

1. a. What is the rational number x pictured below?
   b. What point corresponds to $3 \cdot x$?



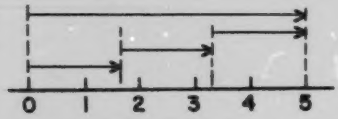
2. a. What is the rational number x pictured below?
   b. What multiplication problem does the diagram show?
   c. What division problem does the diagram show?
   d. What point corresponds to $4 \cdot x$?



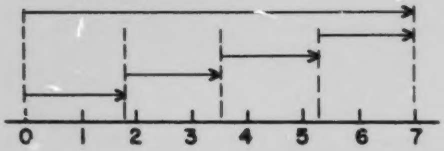
96

3. State the multiplication problem and the corresponding division problem which each diagram represents.

a.

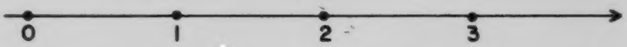


b.

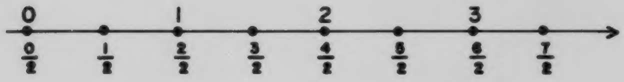


8-3. Comparing Rational Numbers.

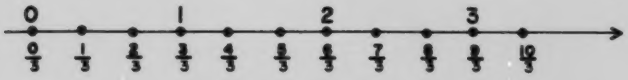
There is another way to locate the rational numbers on the number line. After the counting numbers are located on the number line, you can label other points by dividing each interval between each two successive marked points into halves, thirds, fourths, etc.



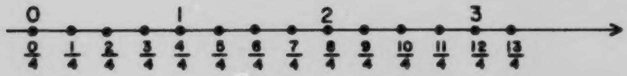
Counting numbers are labeled.


0
1
2
3
0/2
1/2
2/2
3/2
4/2
5/2
6/2
7/2

Multiples of $\frac{1}{2}$ are labeled.

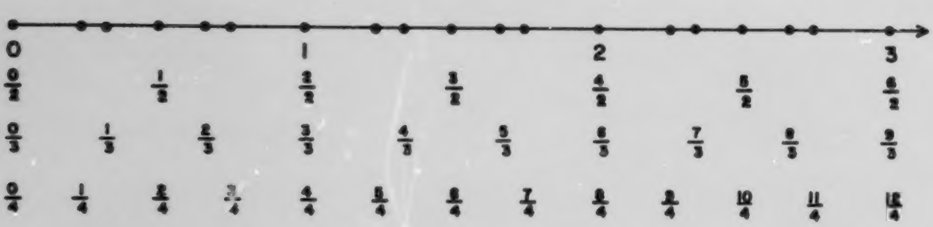

0
1
2
3
0/3
1/3
2/3
3/3
4/3
5/3
6/3
7/3
8/3
9/3
10/3

Multiples of $\frac{1}{3}$ are labeled.


0
1
2
3
0/4
1/4
2/4
3/4
4/4
5/4
6/4
7/4
8/4
9/4
10/4
11/4
12/4
13/4

Multiples of $\frac{1}{4}$ are labeled.

Think of this process of dividing the line as continuing without end. Now put all the points together on one line and you have a labeling of points corresponding to a set of numbers like this.

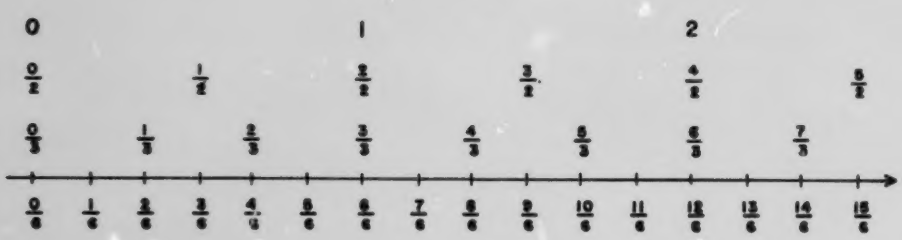

0
1
2
3
0/2
1/2
2/2
3/2
4/2
5/2
6/2
0/3
1/3
2/3
3/3
4/3
5/3
6/3
7/3
8/3
9/3
0/4
1/4
2/4
3/4
4/4
5/4
6/4
7/4
8/4
9/4
10/4
11/4
12/4

(and so on.)

This set of numbers is the set of rational numbers. Each number in the set has a fraction name. Every possible fraction can be attached to a point by this process. Notice that equivalent fractions correspond to the same point. Each rational number corresponds to just one point no matter what fraction we use to name it.

Exercises 8-3a

(Class Discussion)


0
1
2
0/2
1/2
2/2
3/2
4/2
5/2
0/3
1/3
2/3
3/3
4/3
5/3
6/3
7/3
0/6
1/6
2/6
3/6
4/6
5/6
6/6
7/6
8/6
9/6
10/6
11/6
12/6
13/6
14/6
15/6

On the number line above are located numbers written as fractions with denominator 6. These numbers have other names and some of these names are also shown.

1. What is a common name for $\frac{6}{6}$?
2. What is a common name for $\frac{12}{6}$?
3. Which of the numbers $\frac{12}{6}$ and $\frac{6}{6}$ is the greater?
4. Which of the numbers $\frac{12}{6}$ and $\frac{6}{6}$ lies farther to the right on the number line?
5. Which of the numbers $\frac{1}{6}$ and $\frac{2}{6}$ lies farther to the right on the number line?
6. a. Which is the greater number $\frac{1}{6}$ or $\frac{2}{6}$?
   b. Compare $\frac{1}{6}$ and $\frac{2}{6}$ using the symbol $<$.
7. a. Compare $\frac{10}{6}$ and $\frac{9}{6}$ using the symbol $<$.
   b. Which of the numbers $\frac{10}{6}$ and $\frac{9}{6}$ lies farther to the right on the number line?
8. If two fractions have the same denominator how can you tell which represents the greater number?
9. If two different numbers are located on the number line how can you tell which is the greater number?

Two very important facts about comparison of numbers are:

1. If two different fractions have the same denominator, then the fraction with the larger numerator represents the greater number.
2. Of the different numbers on the number line, the number farther to the right is the greater number.

One of the most important properties of the number line is that it preserves the order of the rational numbers. That is, the rational numbers are arranged on the number line from left to right in order of increasing size.

Exercises 8-3b
(Class Discussion)

1. Supply the correct symbol, $>$ or $<$, in each statement below.

Example: $\frac{7}{10}$? $\frac{9}{10}$

Solution: $\frac{7}{10} < \frac{9}{10}$

a. $\frac{9}{6}$? $\frac{10}{6}$

b. $\frac{11}{3}$? $\frac{13}{3}$

c. $\frac{6}{5}$? $\frac{4}{5}$

d. $\frac{8}{7}$? $\frac{11}{7}$

e. $\frac{3}{12}$? $\frac{4}{12}$

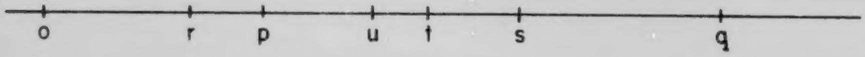
f. $\frac{6}{9}$? $\frac{4}{9}$

g. $\frac{4}{19}$? $\frac{22}{19}$

h. $\frac{13}{24}$? $\frac{14}{24}$

i. $\frac{11}{2}$? $\frac{12}{2}$

2. Certain rational numbers are indicated on the number line.



Supply the correct symbol, $<$ or $>$, in the statements below and give reasons for your answers.

Example: $q > r$ since q is to the right of r on the number line.

a. q ? r

b. p ? s

c. t ? q

d. s ? u

e. s ? q

f. p ? t

g. u ? r

h. r ? t

i. u ? p

Which is the greater number $\frac{3}{2}$ or $\frac{5}{3}$? These rational numbers are expressed as fractions with different denominators, and so far you have only compared numbers expressed as fractions with the same denominator.

If you look back at the diagram for Exercises 8-3a, you see that "$\frac{3}{2}$" is another name for $\frac{9}{6}$ and "$\frac{5}{3}$" is another name for $\frac{10}{6}$.

$$\frac{3}{2} = \frac{9}{6} \qquad \frac{5}{3} = \frac{10}{6}$$

Since you already know

$$\frac{9}{6} < \frac{10}{6}$$

then $\frac{3}{2} < \frac{5}{3}$.

In order to compare $\frac{3}{2}$ and $\frac{5}{3}$ you express the two rational numbers as fractions with the same denominator and then compare these fractions.

Consider another example. Compare $\frac{5}{7}$ and $\frac{2}{3}$. How do you proceed? Try to express these as fractions with the same denominator. To do this, look for a common denominator for $\frac{5}{7}$ and $\frac{2}{3}$. That is, look for a common multiple of 7 and 3. A multiple of 7 and 3 is 21. Find equivalent fractions for $\frac{5}{7}$ and $\frac{2}{3}$ with denominator 21.

$$\frac{5}{7} = \frac{5}{7} \cdot \frac{3}{3} = \frac{15}{21}$$

$$\frac{2}{3} = \frac{2}{3} \cdot \frac{7}{7} = \frac{14}{21}$$

Since $15 > 14$ then $\frac{15}{21} > \frac{14}{21}$ or $\frac{5}{7} > \frac{2}{3}$.

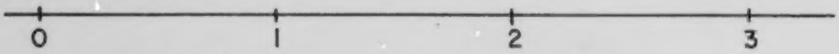
Exercises 8-3c

1. Compare the following pairs of rational numbers by expressing each pair as fractions with the same denominator:

   a. $\frac{3}{4}$, $\frac{5}{9}$ c. $\frac{21}{22}$, $\frac{10}{11}$

   b. $\frac{4}{2}$, $\frac{7}{3}$ d. $\frac{13}{12}$, $\frac{25}{24}$

2. Locate the pairs of numbers in Problem 1 on the number line, by estimating their relative positions.



3. Store A sold candy at 50¢ for $\frac{3}{4}$ of a pound. Store B sold candy at 50¢ for $\frac{7}{8}$ of a pound. At which store is the candy cheaper?

4. It costs $35,000 a mile to build a road. The state will pay $\frac{3}{10}$ of the cost, the county will pay $\frac{4}{15}$ of the cost and the town will pay the rest. Will the state or the county pay more of the total cost of the road?

Now that you have a method for comparing two rational numbers, try to compare

$$\frac{a}{b} \text{ and } \frac{c}{d}.$$

Your first step is to find a common denominator for these two fractions. Since $b \cdot d$ is a multiple of both b and d

$$\frac{a}{b} = \frac{a \cdot d}{b \cdot d} \quad \text{and} \quad \frac{c}{d} = \frac{b \cdot c}{b \cdot d}$$

The numbers $\frac{a}{b}$ and $\frac{c}{d}$ are expressed as fractions with the same denominator. Now you compare the numerators of these fractions, $a \cdot d$ and $b \cdot c$. The three possibilities are expressed below:

Comparison Property: If a, b, c and d are whole numbers with b and d different from 0, then

1. $\frac{a}{b} < \frac{c}{d}$ if $a \cdot d < b \cdot c$
2. $\frac{a}{b} > \frac{c}{d}$ if $a \cdot d > b \cdot c$
3. $\frac{a}{b} = \frac{c}{d}$ if $a \cdot d = b \cdot c$

Example 1. Use this property to compare $\frac{3}{2}$ and $\frac{5}{3}$. Let $a = 3$, $b = 2$, $c = 5$, $d = 3$. Then

$$\frac{a}{b} = \frac{3}{2} \quad \text{and} \quad \frac{c}{d} = \frac{5}{3}.$$

Now $a \cdot d = 3 \cdot 3 = 9$ and $b \cdot c = 2 \cdot 5 = 10$.

Since $9 < 10$ or $a \cdot d < b \cdot c$, the property says that

$$\frac{a}{b} < \frac{c}{d} \quad \text{or} \quad \frac{3}{2} < \frac{5}{3}.$$

Example 2. Compare $\frac{9}{21}$ and $\frac{6}{14}$

Here

$$\frac{a}{b} = \frac{9}{21} \quad \text{and} \quad \frac{c}{d} = \frac{6}{14}$$

$$a \cdot d = 9 \cdot 14 = 126; \quad b \cdot c = 21 \cdot 6 = 126$$

Then $a \cdot d = b \cdot c$ and therefore

$$\frac{a}{b} = \frac{c}{d} \quad \text{or} \quad \frac{9}{21} = \frac{6}{14}.$$

Exercises 8-3d

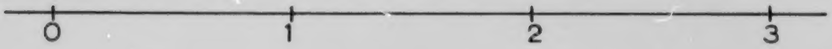
Using the Comparison Property, supply the correct symbol $>$, $<$, or $=$ between the given pairs of numbers. Arrange your work as shown in the example.

Example: $\frac{3}{4}$? $\frac{10}{13}$

Solution: $3 \cdot 13 = 39$, $4 \cdot 10 = 40$, $39 < 40$ so that $\frac{3}{4} < \frac{10}{13}$.

1. $\frac{9}{11} ? \frac{16}{19}$
2. $\frac{15}{4} ? \frac{11}{3}$
3. $\frac{10}{15} ? \frac{8}{12}$
4. $\frac{3}{4} ? \frac{10}{13}$
5. $\frac{15}{4} ? \frac{18}{5}$
6. $\frac{3}{7} ? \frac{5}{12}$
7. $\frac{14}{19} ? \frac{2}{3}$
8. $\frac{4}{7} ? \frac{3}{5}$
9. $\frac{3}{17} ? \frac{1}{5}$
10. $\frac{3}{8} ? \frac{1}{3}$
11. $\frac{4}{3} ? \frac{5}{4}$
12. $\frac{4}{6} ? \frac{6}{9}$
13. $\frac{11}{8} ? \frac{4}{3}$
14. $\frac{7}{4} ? \frac{5}{3}$
15. $\frac{19}{11} ? \frac{7}{4}$
16. $\frac{14}{5} ? \frac{31}{11}$
17. $\frac{6}{8} ? \frac{15}{20}$
18. $\frac{2}{9} ? \frac{1}{5}$
19. $\frac{14}{6} ? \frac{35}{15}$
20. $\frac{9}{7} ? \frac{5}{4}$
21. $\frac{12}{15} ? \frac{8}{10}$
22. $\frac{11}{7} ? \frac{8}{5}$
23. $\frac{6}{10} ? \frac{9}{15}$
24. $\frac{8}{13} ? \frac{5}{8}$

*25. a. On the number line below locate $\frac{3}{2}$, $\frac{4}{3}$, $\frac{9}{5}$ and the reciprocals of these numbers. (Do this very roughly but be sure the order of the numbers is correct.)



b. Replace the question marks by the correct symbol < or >.

$\frac{3}{2} ? \frac{4}{3}$ $\quad$ $\frac{2}{3} ? \frac{3}{4}$ $\quad$ $\frac{2}{3} ? \frac{5}{9}$ $\quad$ $\frac{3}{2} ? \frac{9}{5}$

$\frac{4}{3} ? \frac{9}{5}$ $\quad$ $\frac{3}{4} ? \frac{5}{9}$

c. From Part (b) above replace "?" in the general statement about reciprocals, with the correct symbol, < or >.

If $\frac{a}{b} < \frac{c}{d}$ then $\frac{b}{a} ? \frac{d}{c}$.

8-4. Mixed Numbers.

If the numerator and denominator of a fraction are the same, as in $\frac{6}{6}$, then the fraction is a numeral for 1.

$$\frac{6}{6} = 1$$

If the numerator is less than the denominator, as in $\frac{5}{6}$, then the fraction represents a number which is less than 1.

$$\frac{5}{6} < \frac{6}{6}$$

If the numerator is greater than the denominator, as in $\frac{7}{6}$, then the fraction represents a number greater than 1.

$$\frac{7}{6} > \frac{6}{6}.$$

Fractions in which the numerator is less than the denominator are commonly called proper fractions. Fractions in which the numerator is either greater than or equal to the denominator are called improper fractions. Unfortunately, the name "improper" seems to suggest that there is something wrong with this sort of fraction. This is not really the case. These fractions are just as good as any other fractions. You should notice that proper fractions are names for numbers which are less than 1, while improper fractions are names for numbers which are either equal to 1 or greater than 1.

Exercises 8-4a
(Class Discussion)

1. Tell in each case whether the given fraction is proper or improper .

a. $\frac{7}{5}$

b. $\frac{19}{18}$

c. $\frac{3}{4}$

d. $\frac{6}{2}$

e. $\frac{8}{4}$

f. $\frac{2}{7}$

g. $\frac{7}{2}$

h. $\frac{4}{9}$

i. $\frac{9}{4}$

j. $\frac{33}{34}$

k. $\frac{34}{33}$

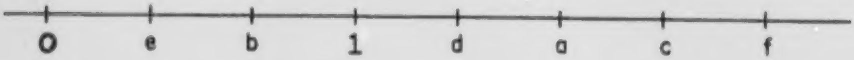
l. $\frac{9}{9}$

m. $\frac{2}{2}$

n. $\frac{2}{5}$

o. $\frac{4}{10}$

2. Certain numbers are located on the number line.



Tell which of these numbers will have names that are proper fractions and which will have names that are improper fractions.

Every improper fraction can be expressed either as a whole number or as the sum of a whole number and a proper fraction. Both of these possibilities are illustrated by these examples

$$\frac{12}{6} = 2$$

$$\frac{13}{6} = \frac{12}{6} + \frac{1}{6} = 2 + \frac{1}{6}$$

It is customary to use the expression

$$2\frac{1}{6}\ .$$

as a short way of writing

$$2 + \frac{1}{6}.$$

Similarly

$$3\frac{5}{7} = 3 + \frac{5}{7}, \quad 13\frac{2}{3} = 13 + \frac{2}{3}.$$

Such expressions as

$$2\frac{1}{6}, \quad 3\frac{5}{7}, \quad 13\frac{2}{3}$$

are commonly called mixed numbers. This is a very bad name because it is not the numbers which are mixed, but the numerals. We know that

$$2\frac{1}{6} = \frac{13}{6},$$

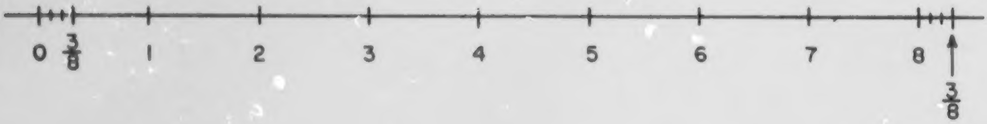
but we do not say that $\frac{13}{6}$ is a mixed number. When we talk about mixed numbers we are talking about the things we write. But we do not write numbers, we write numerals. We see that it might be better to refer to expressions like

$$2\frac{1}{6}, \quad 3\frac{5}{7}, \quad 13\frac{2}{3}$$

as "mixed numerals" rather than "mixed numbers". But we shall speak of "mixed numbers" because this expression is the one commonly used.

It is often convenient to express rational numbers as mixed numbers rather than as improper fractions. This is particularly true when making measurements. For example we could say that a soda straw is $8\frac{3}{8}$ inches long or we could say that it is $\frac{67}{8}$ inches long. The numbers $8\frac{3}{8}$ and $\frac{67}{8}$ are the same. But when we say that the straw is $8\frac{3}{8}$ inches long we have a clearer idea of how long it is. We know at once that it is more than 8 and less than 9 inches long.

We can place this number on the number line more quickly when it is expressed as $8\frac{3}{8}$.



If a woman reading a recipe found that a cake required $\frac{19}{8}$ cups of flour what should she do? She might take 19 cups of flour and split this quantity of flour into 8 equal parts. Or she might measure out $\frac{1}{8}$ cup of flour 19 times. In the first case she was thinking

$$\frac{19}{8} = 19 \div 8.$$

In the second case she was thinking

$$\frac{19}{8} = 19 \cdot \frac{1}{8}$$

Both of these methods are correct but neither is convenient. If the recipe called for $2\frac{3}{8}$ cups of flour (the same amount of flour as above), the woman would probably do the sensible thing. She would measure 2 cups and then measure $\frac{3}{8}$ cup of flour.

To express an improper fraction as a mixed number you may proceed as follows:

$$\frac{17}{6} = \frac{12 + 5}{6} = \frac{12}{6} + \frac{5}{6} = 2 + \frac{5}{6} = 2\frac{5}{6}.$$

The idea is to find the largest multiple of the denominator which is less than the numerator. In this case we saw that 12 was the largest multiple of 6 which is less than 17.

Another way is to divide 17 by 6.

$$\begin{array}{r} 2 \\ 6\overline{)17} \\ \underline{12} \\ 5 \end{array}$$

We say the quotient is 2 and the remainder is 5. This means

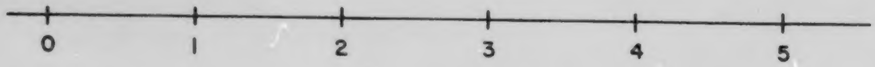
$$17 = 2 \cdot 6 + 5$$

or

$$\frac{17}{6} = 2 + \frac{5}{6} = 2\frac{5}{6}$$

Exercises 8-4b

1. Copy the number line as shown below.



Locate on the number line.

a. $2\frac{3}{4}$	e. $\frac{14}{3}$	i. $4\frac{2}{3}$
b. $4\frac{3}{5}$	f. $\frac{11}{5}$	j. $2\frac{1}{5}$
c. $1\frac{2}{3}$	g. $\frac{7}{2}$	k. $3\frac{1}{2}$
d. $3\frac{1}{3}$	h. $\frac{15}{4}$	l. $3\frac{3}{4}$

2. Find two consecutive whole numbers between which each of the following numbers is located. Use the symbol $<$ in your answer.
(For example: $\frac{21}{8} = 2\frac{5}{8}$ so $2 < \frac{21}{8} < 3$.)

a. $\frac{3}{2}$

b. $\frac{3}{4}$

c. $\frac{8}{3}$

d. $\frac{70}{11}$

e. $\frac{24}{7}$

f. $\frac{81}{8}$

g. $\frac{111}{11}$

h. $\frac{33}{5}$

i. $\frac{84}{9}$

j. $\frac{57}{8}$

3. Express the following improper fractions as mixed numbers.

a. $\frac{7}{6}$

b. $\frac{14}{3}$

c. $\frac{13}{2}$

d. $\frac{9}{4}$

e. $\frac{27}{7}$

f. $\frac{42}{5}$

g. $\frac{65}{3}$

h. $\frac{47}{4}$

i. $\frac{167}{16}$

j. $\frac{435}{14}$

k. $\frac{229}{21}$

l. $\frac{395}{151}$

m. $\frac{409}{408}$

n. $\frac{956}{222}$

o. $\frac{356}{209}$

p. $\frac{555}{77}$

Suppose a rational number is expressed as a mixed number and you wish to express it instead as an improper fraction. You proceed as in the following example.

Express $6\frac{4}{5}$ as an improper fraction.

$$6\frac{4}{5} = 6 + \frac{4}{5} = 6 \cdot \frac{5}{5} + \frac{4}{5} = \frac{30}{5} + \frac{4}{5} = \frac{34}{5}$$

Addition problems with mixed numbers can be done in several ways.

Example 1:

$$\begin{aligned} 6\tfrac{1}{2} + 4\tfrac{1}{3} &= (6 + \tfrac{1}{2}) + (4 + \tfrac{1}{3}) \\ &= (6 + 4) + (\tfrac{1}{2} + \tfrac{1}{3}) \\ &= 10 + (\tfrac{3}{6} + \tfrac{2}{6}) \\ &= 10 + \tfrac{5}{6} \\ &= 10\tfrac{5}{6} \end{aligned}$$

Notice we have used the commutative and associative properties of addition to regroup the "whole number parts" and the "fraction parts" of our mixed numbers. We add these separately and then combine them.

Sometimes the sum of the fraction parts is greater than 1. Then another step is necessary.

Example 2: $2\frac{3}{5} + 4\frac{2}{3} = (2 + 4) + \left(\frac{3}{5} + \frac{2}{3}\right) = 6 + \left(\frac{9}{15} + \frac{10}{15}\right)$

$$= 6 + \frac{19}{15} = 6 + 1\frac{4}{15}$$

$$= (6 + 1) + \frac{4}{15} = 7\frac{4}{15}$$

The work is sometimes arranged like this:

$$\begin{array}{ccc} 2\frac{3}{5} & = & 2\frac{9}{15} \\ \underline{4\frac{2}{3}} & = & \underline{4\frac{10}{15}} \\ & & 6\frac{19}{15} = 7\frac{4}{15} \end{array}$$

Example 3:

When we subtract, this same way of arranging our work is convenient.

$$\begin{array}{ccc} 4\frac{2}{3} & = & 4\frac{10}{15} \\ \underline{2\frac{3}{5}} & = & \underline{2\frac{9}{15}} \\ & & 2\frac{1}{15} \end{array}$$

Example 4:

Sometimes in subtraction problems it is necessary to regroup. Consider

$$4\frac{2}{5} - 2\frac{5}{7}$$

We arrange our work like this:

$$\begin{array}{ccc} 4\frac{2}{5} & = & 4\frac{14}{35} \\ \underline{2\frac{5}{7}} & = & \underline{2\frac{25}{35}} \end{array}$$

Now we see that $\frac{14}{35} < \frac{25}{35}$ so that we cannot subtract as things stand now. We could express both $4\frac{2}{5}$ and $2\frac{5}{7}$ as improper fractions and subtract. Or we can proceed as follows:

$$4\frac{14}{35} = 3 + 1 + \frac{14}{35} = 3 + \frac{35}{35} + \frac{14}{35} = 3\frac{49}{35}$$

(first regroup 4 as 3 + 1) (then express 1 as $\frac{35}{35}$) (and finally combine the last two terms.)

Now we have:

$$\begin{array}{ccc} 4\frac{2}{5} & = & 3\frac{49}{35} \\ - & & \\ \underline{2\frac{5}{7}} & = & \underline{2\frac{25}{35}} \\ & & 1\frac{24}{35} \end{array}$$

It may be convenient to express the mixed numbers as improper fractions before performing the operations.

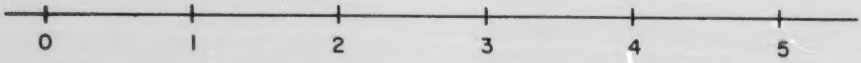
Example 5: $4\frac{2}{3} + 2\frac{3}{5} = \frac{14}{3} + \frac{13}{5} = \frac{70}{15} + \frac{39}{15} = \frac{109}{15} = 7\frac{4}{15}$.

Example 6: $4\frac{2}{3} - 2\frac{3}{5} = \frac{14}{3} - \frac{13}{5} = \frac{70}{15} - \frac{39}{15} = \frac{31}{15} = 2\frac{1}{15}$.

Exercises 8-4c

1. Find the given mixed numbers on the number line below. (by eye).

a. $2\frac{1}{2}$ b. $3\frac{2}{3}$ c. $4\frac{3}{5}$ d. $2\frac{4}{5}$ e. $3\frac{2}{7}$ f. $4\frac{1}{3}$



2. Express the following mixed numbers as improper fractions.

a. $1\frac{3}{4}$
b. $2\frac{1}{2}$
c. $3\frac{2}{3}$
d. $2\frac{3}{5}$
e. $4\frac{3}{5}$
f. $3\frac{2}{7}$
g. $4\frac{1}{3}$
h. $6\frac{5}{9}$
i. $8\frac{3}{8}$
j. $10\frac{5}{7}$
k. $5\frac{5}{12}$
l. $11\frac{3}{5}$
m. $14\frac{2}{7}$
n. $18\frac{1}{8}$
o. $21\frac{1}{21}$
p. $19\frac{1}{19}$

3. Perform the indicated operations.

a. $1\frac{1}{4} + 2\frac{1}{2}$
b. $2\frac{2}{5} + 3\frac{1}{3}$
c. $3\frac{2}{7} + 1\frac{1}{5}$
d. $4\frac{1}{2} - 2\frac{1}{3}$
e. $3\frac{2}{5} - 1\frac{1}{4}$
f. $4\frac{1}{5} + 3\frac{1}{6}$
g. $6\frac{1}{2} - 4\frac{2}{3}$
h. $5\frac{2}{5} - 4\frac{1}{2}$
i. $6\frac{2}{3} + 4\frac{1}{2}$
j. $5\frac{4}{5} + 2\frac{3}{7}$
k. $4\frac{5}{9} - 2\frac{2}{3}$
l. $11\frac{1}{3} - 6\frac{7}{8}$

4. Max had 10 cups of lemonade for a party. He found 3 pitchers. One held $2\frac{1}{4}$ cups, one $3\frac{1}{4}$ cups, and the largest one, $4\frac{3}{4}$ cups. Did the pitchers hold all the lemonade?

5. How far did a group of boys hike on a four day trip, if the distances hiked on the separate days were $6\frac{3}{10}$ miles, $7\frac{9}{10}$ miles, $5\frac{7}{10}$ miles, $6\frac{1}{10}$ miles?

*6. A painter used $2\frac{1}{2}$ gallons of white, 1 quart of green, and 1 pint of blue to mix some paint. How many gallons of paint did he have in the mixture?

In other cases, particularly in multiplication and division problems it is better to express numbers as improper fractions rather than as mixed numbers. You should be familiar with both ways of expressing numbers and be able to switch from one way to the other.

Example 1: $4\frac{2}{3} \cdot 2\frac{3}{5} = \frac{14}{3} \cdot \frac{13}{5} = \frac{14 \cdot 13}{3 \cdot 5} = \frac{182}{15} = 12\frac{2}{15}$

Example 2: $4\frac{2}{3} \div 2\frac{3}{5} = \frac{14}{3} \div \frac{13}{5} = \frac{14}{3} \cdot \frac{5}{13} = \frac{14 \cdot 5}{3 \cdot 13} = \frac{70}{39} = 1\frac{31}{39}$

Exercises 8-4d

1. Perform the indicated operations.
   - a. $2\frac{1}{2} \cdot 3\frac{1}{3}$
   - b. $\frac{3}{4} \cdot 4\frac{1}{3}$
   - c. $5\frac{2}{3} \cdot 3\frac{3}{4}$
   - d. $6\frac{2}{7} \div 2\frac{1}{2}$
   - e. $6\frac{3}{5} \cdot 8\frac{1}{3}$
   - f. $\frac{3}{4} \div 1\frac{3}{4}$
   - g. $4\frac{2}{3} \div 2\frac{4}{5}$
   - h. $3\frac{3}{5} \div 5\frac{1}{2}$
   - i. $\frac{14}{5} \cdot 2\frac{2}{7}$

2. For a school play 16 costumes are needed. Each requires $2\frac{3}{8}$ yards of material. The teacher was able to buy part of a bolt which contained $39\frac{3}{4}$ yards of material.
   - a. Did the teacher get enough material?
   - b. What is the difference between the amount needed and the amount purchased?

3. A delivery clerk is allowed $7\frac{1}{2}$¢ per mile for the use of his car. How much is he paid if he travels $40\frac{4}{5}$ miles?

4. A secretary typed for 6 hours. She finished $\frac{3}{4}$ of her work. How long will it take her to do the whole job?

5. A housewife made 7 jars of jelly from $\frac{1}{3}$ of a crate of berries. How many jars can she make from the whole crate?

6. In the city where Jim lives there are 12 blocks to the mile. It is 9 blocks from his home to school. How many miles does he walk in going to and returning from school?

8-5. Complex Fractions.

You will often see expressions called complex fractions which look like this

$$\frac{\frac{5}{3}}{\frac{2}{7}}.$$

(Notice that the middle line is longer than the other two lines.) This is a numeral in which the numerator and denominator are fractions. What does this mean? Remember that when a and b are whole numbers than

$$\frac{a}{b}$$

stands for the result of dividing a by b. That is

$$\frac{a}{b} = a \div b.$$

In the same way when a and b are fractions, we agree $\frac{a}{b}$ means $a \div b$. So

$$\frac{\frac{5}{3}}{\frac{2}{7}} \qquad \text{means} \quad \frac{5}{3} \div \frac{2}{7}$$

In our definition of a fraction we now include expressions of the form $\frac{a}{b}$ where a and b may be rational numbers instead of just counting numbers. Some examples of fractions are:

$$\frac{2}{\frac{7}{5}}, \qquad \frac{1\frac{1}{2}}{\frac{2}{3}}, \qquad \frac{0.15}{100}, \qquad \frac{1.5}{7.5}$$

The main problem with complex fractions is to learn to express them as fractions where the numerator and denominator are whole numbers. You are able to do this because you know the rational numbers are closed under division.

Simplify: $\frac{\frac{5}{3}}{\frac{2}{7}}$.

$$\frac{\frac{5}{3}}{\frac{2}{7}} = \frac{5}{3} \div \frac{2}{7} = \frac{5}{3} \cdot \frac{7}{2} = \frac{5 \cdot 7}{3 \cdot 2} = \frac{35}{6}$$

The same method applies to fractions like $\frac{5}{\frac{2}{3}}$

$$\frac{5}{\frac{2}{3}} = 5 \div \frac{2}{3} = \frac{5}{1} \cdot \frac{3}{2} = \frac{15}{2}$$

Complex fractions in which the numerator is 1 are particularly interesting. For example

$$\frac{1}{\frac{5}{6}} = 1 \div \frac{5}{6} = 1 \cdot \frac{6}{5} = \frac{6}{5} .$$

And $\frac{6}{5}$ is the reciprocal of $\frac{5}{6}$.

In the same way, if a and b are different from 0 we have

$$\frac{1}{\frac{a}{b}} = 1 \div \frac{a}{b} = 1 \cdot \frac{b}{a} = \frac{b}{a} .$$

And $\frac{b}{a}$ is the reciprocal of $\frac{a}{b}$.

In Chapter 6, we saw that if x is a counting number then $\frac{1}{x}$ is the reciprocal of x. Now we see that if x is any rational number (except 0) then $\frac{1}{x}$ is the reciprocal of x.

Exercises 8-5

1. Simplify.

a. $\dfrac{\frac{2}{3}}{\frac{3}{4}}$

b. $\dfrac{\frac{6}{7}}{\frac{3}{5}}$

c. $\dfrac{\frac{1}{3}}{\frac{1}{5}}$

d. $\dfrac{\frac{2}{5}}{\frac{3}{4}}$

e. $\dfrac{\frac{4}{3}}{\frac{3}{4}}$

f. $\dfrac{\frac{6}{5}}{\frac{3}{10}}$

g. $\dfrac{\frac{4}{9}}{\frac{2}{3}}$

h. $\dfrac{\frac{15}{8}}{\frac{5}{6}}$

i. $\dfrac{\frac{7}{8}}{\frac{9}{10}}$

2. Simplify. (Hint: first express both numerator and denominator as fractions.)

a. $\dfrac{1\frac{1}{2}}{2\frac{1}{4}}$

b. $\dfrac{3\frac{1}{5}}{4\frac{2}{3}}$

c. $\dfrac{2\frac{1}{3}}{2\frac{4}{5}}$

d. $\dfrac{\frac{1}{2}+\frac{1}{3}}{\frac{1}{4}+\frac{1}{5}}$

e. $\dfrac{\frac{2}{3}-\frac{3}{5}}{\frac{3}{4}-\frac{2}{3}}$

f. $\dfrac{\frac{1}{3}+2\frac{1}{2}}{\frac{1}{2}+2\frac{1}{3}}$

8-6. Equivalent Fractions.

A rational number has many fraction names. When you compare or add rational numbers expressed as fractions, you often want to find other fraction names for your numbers. The Comparison Property gives you another way to find equivalent fractions.

Example 1. What is the fraction with denominator 30 that is equivalent to $\frac{2}{5}$?

You can write this problem as

$$\frac{n}{30} = \frac{2}{5}$$

The Comparison Property says $\frac{n}{30} = \frac{2}{5}$ if

$$n \cdot 5 = 30 \cdot 2$$

$$5 \cdot n = 60$$ Commutative property of multiplication

$$n = \frac{60}{5}$$ Definition of a rational number.

$$n = 12$$

(The product $n \cdot 5$ or $5 \cdot n$ is usually written as $5n$.)

Sometimes it is useful to express a simple fraction as a complex fraction.

Example 2. Express $\frac{3}{7}$ as a fraction with denominator 10.

$$\frac{3}{7} = \frac{n}{10}$$

$$30 = 7n$$

$$n = \frac{30}{7}$$

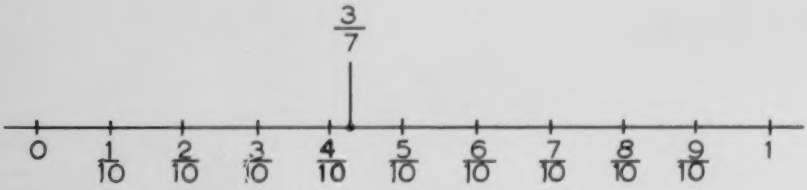
$$n = 4\frac{2}{7}$$

So $\frac{3}{7} = \frac{4\frac{2}{7}}{10}$

Since $\frac{4}{10} < \frac{4\frac{2}{7}}{10} < \frac{5}{10}$

You know $\frac{4}{10} < \frac{3}{7} < \frac{5}{10}$

Having expressed $\frac{3}{7}$ in the form $\frac{4\frac{2}{7}}{10}$ you can locate $\frac{3}{7}$ properly between tenths, on the number line.



Exercises 8-6

a) Use the Comparison Property to write the following as statements about n without fractions

Example: if $\frac{2}{3} = \frac{4}{n}$

Then $2n = 12$

b) Then use the definition of a rational number to find n in simplest form.

$$\text{If } 2n = 12$$
$$\text{then } n = \frac{12}{2}$$
$$\text{or } n = 6\,.$$

1. $\frac{3}{4} = \frac{n}{6}$ 2. $\frac{n}{18} = \frac{1}{6}$ 3. $\frac{2}{7} = \frac{8}{n}$

4. $\frac{n}{10} = \frac{10}{3}$ 5. $\frac{10}{n} = \frac{10}{3}$ 6. $\frac{10}{n} = \frac{3}{10}$ 7. $\frac{10}{10} = \frac{n}{3}$

8. Find a fraction equivalent to $\frac{1}{8}$ which has 100 as the denominator.

9. Find fractions equivalent to $\frac{1}{3}$ which have denominators 10, 100, and 1000.

10. Express the following as fractions with denominator 10.

a. $\frac{1}{9}$ d. $\frac{21}{4}$ g. $\frac{7}{3}$

b. $\frac{2}{3}$ e. $\frac{4}{5}$ h. $\frac{3}{2}$

c. $\frac{4}{7}$ f. $\frac{1}{6}$

11. Locate the points in Problem 10 on a number line to the nearest tenth of a unit.

12. a. Express $\frac{1}{9}$ as a fraction with denominator 10^2.

b. Express $\frac{2}{3}$ as a fraction with denominator 10^3.

c. Express $\frac{21}{4}$ as a fraction with denominator 10^4.

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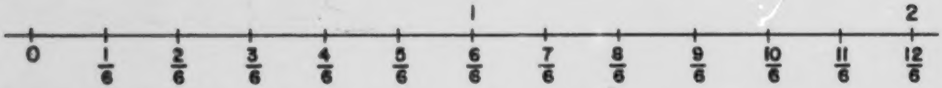
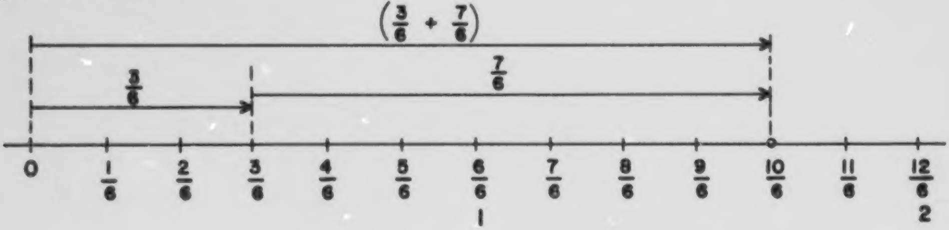
*8-7. Operations on the Number Line.

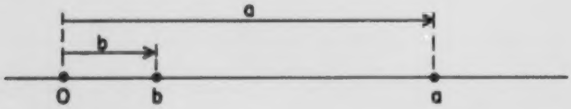
Now that we can locate rational numbers on the number line they can be added and subtracted geometrically in the same way as whole numbers.


0
$\frac{1}{6}$
$\frac{2}{6}$
$\frac{3}{6}$
$\frac{4}{6}$
$\frac{5}{6}$
$\frac{6}{6}$
1
$\frac{7}{6}$
$\frac{8}{6}$
$\frac{9}{6}$
$\frac{10}{6}$
$\frac{11}{6}$
$\frac{12}{6}$
2

Here is a number line with multiples of $\frac{1}{6}$ located on it. Suppose we wish to add $\frac{3}{6}$ and $\frac{7}{6}$. We may add these just as we added whole numbers. The following diagram shows the procedure.

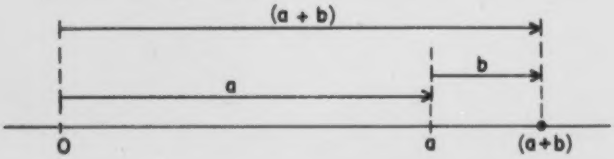

$(\frac{3}{6} + \frac{7}{6})$
$\frac{3}{6}$
$\frac{7}{6}$
0
$\frac{1}{6}$
$\frac{2}{6}$
$\frac{3}{6}$
$\frac{4}{6}$
$\frac{5}{6}$
$\frac{6}{6}$
1
$\frac{7}{6}$
$\frac{8}{6}$
$\frac{9}{6}$
$\frac{10}{6}$
$\frac{11}{6}$
$\frac{12}{6}$
2

The point which corresponds to the number $(\frac{3}{6} + \frac{7}{6})$ is labeled in this example as $\frac{10}{6}$.



Here are two rational numbers a and b located on the number line. The arrows which represent them both start at 0 and end at the point which corresponds to the number.

To find the point on the number line which corresponds to $(a + b)$ we proceed as shown below

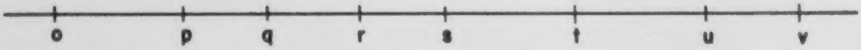


1. The arrow representing a starts at 0 and ends at a.
2. The arrow representing b has been transferred to the end of "arrow a".
3. The arrow from 0 to the end of "arrow b" represents the sum "a + b", and locates the point on the number line which corresponds to this sum.

The same type of diagram can be used for subtraction.

Exercises 8-7

Here is the number line with certain rational numbers indicated. Answer the questions below. (Use a strip of paper to lay off distances.)



Which of the indicated numbers is:

1. p + t
2. q + t
3. t - p
4. t - q
5. r + q
6. s + p
7. t - s
8. u - p
9. t - r
10. v - t
11. u - t
12. v - q
13. t + o
14. o + s
15. v - o

8-8. Summary.

In this chapter you learned to place any rational number on the number line. You learned how to add and subtract, multiply and divide geometrically. The method of placing numbers on the number line preserves the natural order of the numbers. That is, if $a < b$ then a is to the left of b on the number line.

> The Comparison Property is a rule for telling which of two rational numbers is the larger when they are both expressed as fractions. One important part of this rule gives us a quick way of telling when two numbers are equal. That is $\frac{a}{b} = \frac{c}{d}$ if $a \cdot d = b \cdot c$.

Fractions in which the numerator is less than the denominator are called proper fractions. Fractions in which the numerator is equal to or greater than the denominator are called improper fractions. Therefore proper fractions are names for numbers which are less than 1 while improper fractions are names for numbers which are greater than or equal to 1. Mixed numbers are numerals such as $2\frac{3}{7}$. This last number is short for $2 + \frac{3}{7}$. Mixed numbers are a way of expressing rational numbers as the sum of a whole number and a fraction less than 1. Mixed numbers are better than improper fractions for making measurements or for quickly estimating the size of the number. Improper fractions are easier than mixed numbers in multiplication and division problems.

Complex fractions are fractions in which the numerator or the denominator or both are themselves fractions. The complex fraction $\frac{\frac{2}{3}}{\frac{5}{7}}$ stands for the same number as $\frac{2}{3} \div \frac{5}{7}$.

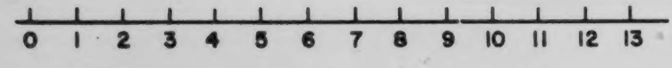
To simplify a complex fraction we perform the indicated division and express the number as an ordinary fraction.

8-9. Chapter Review.

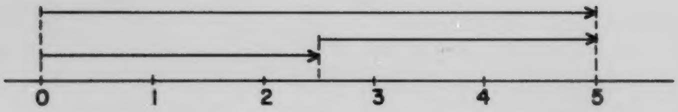
Exercises 8-9

1. On the number line below show how to find

   a. $4 \cdot 3$ c. $5 + 2$
   b. $12 \div 4$ d. $7 - 5$

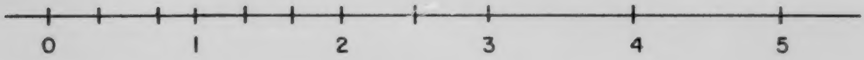


2. The following diagram shows a problem on the number line. State
   a. the addition problem it represents.
   b. the subtraction problem it represents.
   c. the multiplication problem it represents.
   d. the division problem it represents.

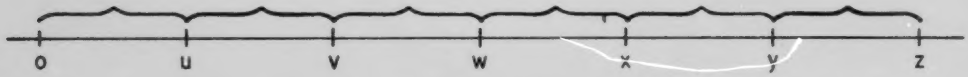


3. Label the indicated points on the number line with the given numerals.

   $\frac{4}{3}$, $\frac{5}{2}$, $\frac{1}{3}$, $\frac{5}{3}$, $\frac{3}{4}$



4. The segments indicated by the braces are equal in length.



   u, v, w, x, y, z are numbers on the number line. Which of the indicated numbers is equal to:

   a. $w + v$
   b. $z - x$
   c. $2 \cdot v$
   d. $3 \cdot v$
   e. $\frac{1}{2} \cdot x$
   f. $\frac{3}{4} \cdot x$
   g. $\frac{2}{3} \cdot z$
   h. $\frac{4}{3} \cdot w$
   i. $\frac{5}{2} \cdot v$
   j. $w \div 3$
   k. $v \div \frac{1}{2}$
   l. $x \div \frac{2}{3}$

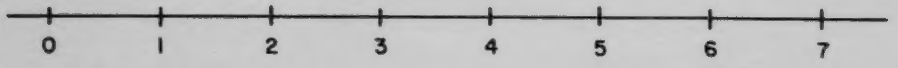
5. On the number line below, locate

   a. the number 6 and its reciprocal.

   b. the number $\frac{3}{4}$ and its reciprocal.

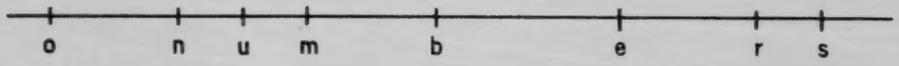
   c. the number $2\frac{1}{2}$ and its reciprocal.

   d. the number $\frac{1}{5}$ and its reciprocal.



6. In Problem 5 you can see that in each case the number 1 lies between the given number and its reciprocal. Can you tell why the number 1 will always lie between a given number and its reciprocal?

7. Certain rational numbers are indicated on the number line.



Replace the question marks by the correct symbol, $>$ or $<$.

a. b ? e	d. n ? r
b. e ? m	e. r ? s
c. m ? n	f. s ? u

8. Compare the following pairs of numbers by expressing them as fractions with the same denominator.

a. $\frac{2}{3} ? \frac{3}{4}$	d. $\frac{6}{9} ? \frac{4}{6}$
b. $\frac{3}{6} ? \frac{2}{4}$	e. $\frac{4}{11} ? \frac{1}{3}$
c. $\frac{3}{7} ? \frac{2}{5}$	f. $\frac{16}{7} ? \frac{25}{11}$

9. Compare the following pairs of numbers using the Comparison Property.

a. $\frac{3}{8} ? \frac{2}{5}$

b. $\frac{8}{12} ? \frac{10}{15}$

c. $\frac{13}{7} ? \frac{15}{8}$

d. $\frac{2}{9} ? \frac{3}{10}$

e. $\frac{6}{5} ? \frac{7}{6}$

f. $\frac{6}{7} ? \frac{5}{6}$

10. $\dfrac{1\frac{2}{5}}{4\frac{2}{3}} = ?$

11. $\dfrac{2\frac{1}{2}}{\frac{1}{2}+\frac{3}{4}} = ?$

12. $\dfrac{1\frac{4}{5}}{1\frac{1}{2} + 5\frac{1}{4}} = ?$

13. BRAINBUSTER.

a. Simplify $1 + \dfrac{1}{1 + \frac{1}{2}}$

b. Simplify $1 + \dfrac{1}{1 + \dfrac{1}{1 + \frac{1}{2}}}$

c. Simplify $1 + \dfrac{1}{1 + \dfrac{1}{1 + \dfrac{1}{1 + \frac{1}{2}}}}$

8-10. Cumulative Review.

Exercises 8-10

Complete the statements in Problems 1-5 below.

1. Zero is the identity for __________.

2. Dividing by 10 is the __________ of multiplying by 10.

3. The following shows a one-to-one correspondence between the ____________ numbers and the ______ __________ numbers.

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & \cdots & n & \cdots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \\ 2 & 4 & 6 & 8 & \cdots & 2n & \cdots \end{array}$$

4. The inverse operation of adding 5 is ________ 5.

5. The number of counting numbers between 4 and 5 is ________.

6. In each case, what is the set that is the intersection of the given sets?

   a. The set of even numbers less than 25 and the set of multiples of 3.

   b. The set of baseball players now in the National League and the set of baseball players now in the American League.

7. List the multiples of 4 which are greater than 30 and less than 50.

8. What is the least common multiple of 2, 5, and 7?

9. Perform the indicated operations and simplify.

   a. $\left(\frac{1}{3} + \frac{1}{2}\right) + \frac{1}{6}$

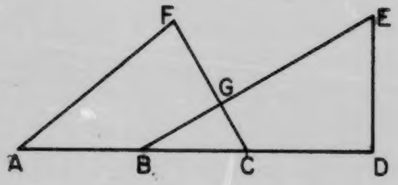
   b. $\left(\frac{7}{10} + \frac{2}{15}\right) - \frac{1}{3}$

   c. $\frac{3}{4} \div \frac{2}{3}$

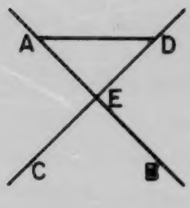
10. Write the reciprocals of the given numbers.

a.	7	d.	$\frac{6}{8931}$
b.	19	e.	$\frac{8}{21}$
c.	$\frac{7}{92}$	f.	$\frac{1}{7}$

11. In the diagram at the right
    a. Name three triangles.
    b. What is $\overleftrightarrow{CF} \cap \triangle EBD$?
    c. What is $\overline{AF} \cap \overline{BE}$?
    d. What is $\overrightarrow{BG} \cup \overrightarrow{BC}$?
    e. Name a point on the B-side of $\overleftrightarrow{FC}$.
    f. What is $\overline{AC} \cup \overline{BD}$?



12. Explain why $\overleftrightarrow{AD}$ is in the plane CEB.



Chapter 9

DECIMALS

9-1. Decimal Notation.

There is a special notation, called decimal notation, used for expressing proper fractions with denominators which are powers of ten. For example:

$\frac{7}{10}$ is written as .7 (read "seven tenths")

$\frac{23}{100}$ is written as .23 (read "twenty-three hundredths")

$\frac{475}{1000}$ is written as .475 (read "four hundred seventy-five thousandths")

Note that the placement of one digit after the decimal point (such as .7) provides a short way of writing a fraction with denominator 10; two places after the decimal point (such as .23) provides a short way of writing a fraction with denominator 100, and so on.

We can also write mixed numbers as decimals. For example:

$1\frac{3}{10}$ is written as 1.3

$12\frac{73}{100}$ is written as 12.73

The point or dot that occurs in the notation "1.3" or "12.73" is called the decimal point. The decimal point always appears to the right of the units place and separates the whole number places from what we call the decimal places:

$$\underbrace{12}_{\text{whole number places}}\underset{\uparrow\atop\text{decimal point}}{.}\underbrace{73}_{\text{decimal places}}$$

Look at 12.73 in greater detail:

$$\begin{aligned} 12.73 &= 12\tfrac{73}{100} \\ &= 12 + \frac{73}{100} \\ &= 12 + \frac{70+3}{100} \\ &= 12 + \frac{70}{100} + \frac{3}{100} \\ &= 12 + \frac{7}{10} + \frac{3}{100} \\ &= 12 + \frac{7}{10} + \frac{3}{(10)^2} \end{aligned}$$

The fact that

$$12.73 = 12 + \frac{7}{10} + \frac{3}{(10)^2}$$

shows us that the first digit after the decimal point still represents a number of tenths (just as it would in 12.7 or in 0.7), and the second digit after the decimal point represents a number of hundredths.

Exercises 9-1a
(Class Discussion)

1. Write in decimal notation:

a.	$\frac{3}{10}$	d.	$1\frac{9}{10}$
b.	$\frac{27}{100}$	e.	$25\frac{7}{10}$
c.	$\frac{389}{1000}$	f.	$81\frac{35}{100}$

2. Read each of the following:

a.	.4	d.	3.47
b.	32.7	e.	.257
c.	.28	f.	17.935

3. Express each of the following decimals as fractions with denominators which are powers of 10.

   a. .55
   b. .21
   c. .4
   d. .013
   e. .105
   f. .012
   g. .03
   h. .84
   i. .064
   j. .25

4. Express each of the following as fractions with denominators which are powers of 10.

   a. Seventy-five hundredths
   b. Five tenths
   c. Thirty hundredths
   d. Eighteen thousandths
   e. Thirty-six hundredths

5. Complete the following:

   a. $5.23 = 5 + \frac{?}{10} + \frac{?}{10^2}$

   b. $3.79 = 3 + \frac{?}{10} + \frac{?}{10^2}$

   c. $28.05 = 28 + \frac{?}{10} + \frac{?}{10^2}$

   d. $75.91 = ? + \frac{?}{10} + \frac{?}{10^2}$

6. Write in decimal notation:

   a. $\frac{3}{10} + \frac{8}{10^2}$

   b. $\frac{7}{10} + \frac{9}{10^2}$

   c. $5 + \frac{0}{10} + \frac{1}{10^2}$

   d. $23 + \frac{9}{10} + \frac{5}{10^2}$

Any fraction with a denominator which is a power of 10 can be written in decimal notation. Here is another example:

$479\frac{532}{1000} = 479.532$ (read "four hundred seventy-nine and five hundred thirty-two thousandths")

We can write $479\frac{532}{1000}$ as follows:

$$\begin{aligned} 479 + \frac{532}{1000} &= 479 + \frac{500 + 30 + 2}{1000} \\ &= 479 + \frac{500}{1000} + \frac{30}{1000} + \frac{2}{1000} \\ &= 479 + \frac{5}{10} + \frac{3}{100} + \frac{2}{1000} \\ &= 479 + \frac{5}{10} + \frac{3}{(10)^2} + \frac{2}{(10)^3} \end{aligned}$$

Now let us see how this fits into our scheme of expanded notation:

$$\begin{aligned} 479.532 &= 400 + 70 + 9 + \frac{5}{10} + \frac{3}{100} + \frac{2}{1000} \\ &= 4(10^2) + 7(10) + 9(1) + 5(\frac{1}{10}) + 3(\frac{1}{10^2}) + 2(\frac{1}{10^3}) \end{aligned}$$

When the number is written as 479.532 it is in positional notation or decimal notation.

Look at the numerals in the parentheses in the last expansion:

$$10^2 \qquad 10 \qquad 1 \qquad \frac{1}{10} \qquad \frac{1}{10^2} \qquad \frac{1}{10^3}$$

Note that as we move from left to right we get the next number by multiplying the preceding one by $\frac{1}{10}$. For example,

$$10 = \frac{1}{10} \cdot 10^2 \; ; \quad 1 = \frac{1}{10} \cdot 10 \; ; \quad \frac{1}{10} = \frac{1}{10} \cdot 1 \; ;$$

$$\frac{1}{10^2} = \frac{1}{10} \cdot \frac{1}{10} \; ; \quad \frac{1}{10^3} = \frac{1}{10} \cdot \frac{1}{10^2}$$

We see that our decimal representation of 479.532 is a natural extension of our place value system of numeration (base ten).

The following chart shows the place values both to the left of and to the right of the units place.

PLACE VALUE CHART

Hundred thousand	Ten thousand	Thousand	Hundred	Ten	Unit	Tenth	Hundredth	Thousandth	Ten-thousandth	Hundred-thousandth
100,000	10,000	1,000	100	10	1	0.1	0.01	0.001	0.0001	0.00001
10^5	10^4	10^3	10^2	10	1	$\frac{1}{10}$	$\frac{1}{10^2}$	$\frac{1}{10^3}$	$\frac{1}{10^4}$	$\frac{1}{10^5}$

Here are several examples of how we change from one notation to another.

Example 1: Write $5(10^2) + 7(10) + 0(1) + 3(\frac{1}{10}) + 2(\frac{1}{10^2})$ in decimal notation.

Answer: 570.32

Example 2: Write 42.356 in expanded form.

Answer: $4(10) + 2(1) + 3(\frac{1}{10}) + 5(\frac{1}{10^2}) + 6(\frac{1}{10^3})$

Exercises 9-1b

1. Write each of the following in decimal notation:

   a. $4(10^3) + 9(10^2) + 6(10) + 7$

   b. $5(10^3) + 6(10^2) + 1(10) + 8 + 3(\frac{1}{10})$

   c. $2(10^2) + 4(10) + 5 + 6(\frac{1}{10}) + 1(\frac{1}{10^2})$

   d. $8(10^2) + 0(10) + 4 + 3(\frac{1}{10}) + 5(\frac{1}{10^2}) + 9(\frac{1}{10^3})$

   e. $5 + 2(\frac{1}{10}) + 4(\frac{1}{10^2})$

   f. $4(\frac{1}{10}) + 8(\frac{1}{10^2}) + 3(\frac{1}{10^3})$

   g. $3(10^2) + 7(10) + 2 + 0(\frac{1}{10}) + 6(\frac{1}{10^2})$

   h. $2(\frac{1}{10^2}) + 0(\frac{1}{10^3}) + 6(\frac{1}{10^4})$

   i. $3 + 5(\frac{1}{10}) + 0(\frac{1}{10^2}) + 7(\frac{1}{10^3})$

2. Write each of the following in expanded form:

   a. 7862

   b. 437.9

   c. 28.64

   d. 347.15

   e. 96.372

   f. 2.465

   g. .384

   h. .013

   i. 24.09

   j. .0039

3. Write each of the following in words. For example 3.001 is written "three and one thousandth".

   a. 658

   b. 3.2

   c. 4.73

   d. 58.29

   e. 759.6

   f. 48.07

   g. 3.209

   h. 37.0106

4. Write in decimal form:
   - a. Five and fifty-two hundredths
   - b. Seven hundred sixty-two and nine tenths
   - c. Three hundred and fifty-two hundredths
   - d. Fourteen hundredths
   - e. Two and seven thousandths
   - f. Sixty and seven hundredths

5. In the following numerals, indicate the number represented by the underlined digit:
   - a. $4\underline{7}8$
   - b. $3\underline{2}59$
   - c. $6\underline{2}$
   - d. $\underline{5}017$
   - e. $21\underline{0}3$
   - f. $1\underline{5}67$

*6. Write $.1_{\text{two}}$ in base 10.

*7. Complete: $.5 = \_\_\_\__{\text{twelve}}$.

*8. Complete: $\frac{5}{6} = .\_\_\_\__{\text{twelve}}$.

*9. Change 10.011_{two} to base 10.

9-2. Arithmetic Operations With Decimals.

In order to be able to use decimals in solving problems we must learn how to add, subtract, multiply, and divide with decimals.

Suppose we wish to add two decimals, for example, $0.73 + 0.84$. We know how to add these numbers without the decimal places: $73 + 84 = 157$. How do we handle the decimal places?

We proceed to add 0.73 and 0.84 as follows:

$$0.73 = 73 \times \frac{1}{100} \quad \text{and} \quad 0.84 = 84 \times \frac{1}{100}$$

Therefore,

$$0.73 + 0.84 = (73 \times \frac{1}{100}) + (84 \times \frac{1}{100})$$

$$= (73 + 84) \times \frac{1}{100}$$

$$= 157 \times \frac{1}{100}$$

$$= 1.57$$

Notice that we use the distributive property here.

Suppose that we wish to add a two decimal place number and a three decimal place number, for example, 0.73 + 0.125.

$$0.73 = 73 \times \frac{1}{100} = 730 \times \frac{1}{1000}$$

and $$0.125 = 125 \times \frac{1}{1000}.$$

We write $73 \times \frac{1}{100}$ as $730 \times \frac{1}{1000}$, so that the factor $\frac{1}{1000}$ appears in both terms of the following sum:

$$0.73 + 0.125 = (730 \times \frac{1}{1000}) + (125 \times \frac{1}{1000})$$

$$= (730 + 125) \times \frac{1}{1000}$$

$$= 855 \times \frac{1}{1000}$$

$$= 0.855$$

These examples can be handled more conveniently by writing one numeral below the other as follows:

$$\begin{array}{r} .73 \\ +\underline{\ .84} \\ 1.57 \end{array} \qquad\qquad \begin{array}{r} 0.73 \\ +\ \underline{0.125} \\ 0.855 \end{array}$$

You notice that a zero is often used in the units place when you see decimals of numbers between 0 and 1, such as in 0.73 and 0.84. This is not necessary and these numbers may also be represented as .73 and .84. The reason for using the zero is to emphasize the location of the decimal point, which might otherwise be overlooked.

Subtraction of decimals can be handled in a similar manner. Consider the problem $0.84 - 0.53$. We proceed as follows:

$$\begin{aligned} 0.84 - 0.53 &= (84 \times \frac{1}{100}) - (53 \times \frac{1}{100}) \\ &= (84 - 53) \times \frac{1}{100} \\ &= 31 \times \frac{1}{100} \\ &= .31 \end{aligned}$$

This example can be handled more conveniently by writing one numeral below the other and subtracting:

$$\begin{array}{r} 0.84 \\ - \underline{0.53} \\ 0.31 \end{array}$$

Here are several additional examples:

$$\begin{array}{r} 0.83 \\ - \underline{0.74} \\ 0.09 \end{array} \qquad\qquad \begin{array}{r} 1.03 \\ - \underline{.25} \\ 0.78 \end{array}$$

Exercises 9-2a

1. Add:

   a. $.3 + .5$
   b. $0.73 + 0.59$
   c. $.6 + 0.85$
   d. $0.719 + 0.382$
   e. $1.0023 + 0.00102$
   f. $1.05 + 0.75 + 21.5$
   g. $23.04 + 9.6 + 16.58$

2. Subtract:
   - a. .9 - .3
   - b. .84 - .76
   - c. .35 - .09
   - d. 1.36 - .97
   - e. 0.625 - 0.550
   - f. 0.500 - 0.125
   - g. 1.005 - 0.0005

3. There are 16 ounces in 1 pound. Which is heavier, 7 ounces or 0.45 lb.?

4. In a mile relay race, four men each run a quarter-mile. Their times are 48.3 seconds, 47.9 seconds, 49.0 seconds, and 48.5 seconds. The meet record is 3 minutes 8.5 seconds. By how many seconds did this team fail to match the record?

5. At the start of a trip a car speedometer read 3827.4 miles. At the end of the trip the reading was 4013.2 miles. How many miles were traveled on the trip?

6. The rainfall in a certain city for a year was 30.04 inches. The rainfall record by months was as follows:

   January - 1.39 inches   April - 4.86 inches
   February- 0.92 inches   May - 3.49 inches
   March - 3.14 inches   June - 1.97 inches

   How much rain fell during the rest of the year?

7. In Problem 6, how much more rainfall was recorded in March than in January?

*8. Add $10.01_{two} + 1.01_{two}$ and then express the answer in the base 10.
Hint: In the base two: $0.1_{two} = \frac{1}{2}$ and $0.01_{two} = \frac{1}{4}$

Suppose we wish to multiply two numbers in decimal form, for example, 0.3×0.25. We know how to multiply these numbers without the decimal places: $3 \times 25 = 75$.

Just as before, we write

$$\begin{aligned} 0.3 \times 0.25 &= (3 \times \frac{1}{10}) \times (25 \times \frac{1}{100}) \\ &= (3 \times 25) \times (\frac{1}{10} \times \frac{1}{100}) \\ &= 75 \times \frac{1}{1000} \\ &= 0.075 \end{aligned}$$

1. How many digits are there to the right of the decimal point in 0.3?
2. How many digits are there to the right of the decimal point in 0.25?
3. What is the sum of the answers to (1) and (2)?
4. How many digits are there to the right of the decimal point in 0.075?
5. Compare the answers to (3) and (4).

Now multiply 0.42×0.29. What is your answer? Answer the five questions above for these numbers. Do the answers to (3) and (4) still agree?

To find the number of decimal places when two numbers are multiplied, add the number of decimal places in the two numerals.

For example, suppose we wish to multiply .735 by .25. The first numeral has three decimal places and the second has two, so there will be five places in the answer. We multiply 735×25, and then mark off five decimal places in the answer, counting from right to left.

$$\begin{array}{r} .735 \\ \underline{.25} \\ 3675 \\ \underline{1470} \\ .18375 \end{array}$$

Thus $.735 \times .25 = .18375$.

Exercises 9-2b

1. Find the products:
   - a. .8 × .7
   - b. .06 × .9
   - c. .05 × .03
   - d. .4 × .004
   - e. .02 × .007
   - f. .8 × 3
   - g. .8 × .09
   - h. .007 × .09
   - i. .6 × 9
   - j. .8 × .006

2. Find the products:
   - a. 1.3 × 2
   - b. 2.5 × .3
   - c. .02 × 1.8
   - d. 1.5 × 1.5
   - e. 2.03 × .7
   - f. .23 × .58
   - g. .723 × .37
   - h. 2.56 × .15
   - i. 30.2 × 6.8
   - j. 12.7 × .49

3. A gallon of water weighs about 8.35 pounds. What is the weight of 4 gallons?

4. There are 4.75 fluid ounces in a can of frozen juice. Three cans of water must be added to this to make a mixture. How many ounces are there in the final mixture?

5. In a certain city the average monthly rainfall is 1.3 inches. At this rate what is the total yearly rainfall?

6. A family pays $83.50 rent per month. How much do they pay per year?

7. If there are 16.5 feet in a rod, how many feet are there in 15 rods?

8. About how many miles is a distance of 4.5 kilometers? (One kilometer is about .6 of a mile.)

Next we wish to examine the operation of division of decimals. First consider the problem of writing a fraction in decimal form. Let us write $\frac{5}{13}$ as a decimal to the thousandths place.

Using the Comparison Property we have:

$$\frac{5}{13} = \frac{x}{1000}$$

$$13x = 5000$$

$$x = 384\tfrac{8}{13}$$

$$\begin{array}{r} 384 \\ 13\overline{)5000} \\ \underline{39} \\ 110 \\ \underline{104} \\ 60 \\ \underline{52} \\ 8 \end{array}$$

Thus $\frac{5}{13} = \frac{384\frac{8}{13}}{1000}$ which leads to an approximate decimal representation for $\frac{5}{13}$:

$$\frac{384}{1000} < \frac{384\frac{8}{13}}{1000} < \frac{385}{1000}$$

In other words, $\frac{5}{13}$ is between $.384$ and $.385$. However, $\frac{8}{13} > \frac{1}{2}$ so that $\frac{5}{13}$ is closer to $.385$ than it is to $.384$. Therefore we write:

$$\frac{5}{13} \approx .385 \qquad (\tfrac{5}{13} \text{ "is approximately equal to" } .385)$$

Now $\frac{5}{13}$ is another way of writing $5 \div 13$. Thus $5 \div 13$ is approximately equal to .385 to the thousandths place.

Let us see what we really did when we used the Comparison Property to divide 5 by 13. First we multiplied 5 by 1000 to obtain 5000. We then divided 5000 by 13 to get $384\frac{8}{13}$. Finally we divided by 1000:

$$\frac{384\frac{8}{13}}{1000} \approx .385.$$

In other words:

$$\frac{5}{13} \times \frac{1000}{1000} = (\frac{5}{13} \times 1000) \div 1000$$

$$= (\frac{5000}{13}) \div 1000$$

$$= (384\frac{8}{13}) \div 1000$$

$$= \frac{384\frac{8}{13}}{1000}$$

We can simplify this process considerably by writing 5 as 5.000. Do the division as if the decimal point were not there. Then place a decimal point in the quotient so that it is directly above the decimal point in the dividend. Divide by 13. As before we find $5 \div 13 \approx .385$.

$$\begin{array}{r} .384 \\ 13\overline{)5.000} \\ \underline{3\,9} \\ 1\,10 \\ \underline{1\,04} \\ 60 \\ \underline{52} \\ 8 \end{array}$$

We can extend this process to all division problems involving decimals. First express the problem in fractional form. Then use the Multiplication Property of 1 to change the denominator to a whole number, and divide. Here is the procedure for finding the quotient $.432 \div 1.35$ to the hundredths place.

Express as a fraction. $\frac{.432}{1.35}$

Multiply by $1 = \frac{100}{100}$. $\frac{.432}{1.35} \times \frac{100}{100} = \frac{43.2}{135}$

Annex a zero, since we want our answer in hundredths, and divide.

$$\begin{array}{r} .32 \\ 135\overline{)43.20} \\ \underline{40\,5} \\ 2\,70 \\ \underline{2\,70} \end{array}$$

As a check, note:

$$\begin{array}{ccccc} (\text{divisor}) & \times & (\text{quotient}) & = & (\text{dividend}) \\ 135 & \times & .32 & = & 43.2 \end{array}$$

Whenever the divisor is a whole number, the dividend and the quotient have the same number of decimal places. By placing the decimal point of the quotient directly above that of the dividend, we locate the decimal point of the quotient automatically in the correct place. This is the reason we want our divisor to be a whole number.

Here is one more example: Find the quotient $2.65 \div 12.3$ to the thousandths place.

$$\begin{aligned} 2.65 \div 12.3 &= \frac{2.65}{12.3} \\ &= \frac{2.65}{12.3} \times \frac{10}{10} \quad \text{(Why do we multiply by } \tfrac{10}{10}\text{?)} \\ &= \frac{26.5}{123} \end{aligned}$$

We want the answer to the thousandths place. Therefore we write 26.5 as 26.500 and divide:

```
       .215
123 |26.500
     24 6
     ----
      1 90
      1 23
      ----
        670
        615
        ---
         55
```

The remainder, 55, is less than one-half of 123. Therefore, to the thousandths place, $2.65 \div 12.3 \approx .215$.

Exercises 9-2c

1. Use the Comparison Property and express each of the following as a fraction with a denominator of 100.

   a. $\frac{1}{7}$

   b. $\frac{2}{3}$

   c. $\frac{1}{9}$

   d. $\frac{4}{3}$

   e. $\frac{10}{9}$

2. Write each of the fractions in Problem 1 as a fraction with a denominator of 1000.

3. Write each of the fractions in Problem 1 as a decimal to the hundredths place.

4. Write each of the fractions in Problem 1 as a decimal to the thousandths place.

5. Find the following quotients:

   a. 0.009 ÷ 0.3
   b. 0.24 ÷ 6
   c. 0.015 ÷ 0.5
   d. 0.532 ÷ 0.04
   e. 2.04 ÷ 0.008
   f. 24.6 ÷ 0.12
   g. 7.35 ÷ 3.5
   h. .756 ÷ 0.42
   i. 9.84 ÷ 4.1
   j. 8.125 ÷ 3.25

6. Find the following quotients to the hundredths place:

   a. 4.32 ÷ 1.2
   b. 72.8 ÷ 0.43
   c. 9.87 ÷ 0.37
   d. 42.6 ÷ 3.02
   e. 798 ÷ 5.8
   f. 27.96 ÷ 2.5
   g. 8.375 ÷ 0.12
   h. 25 ÷ 13
   i. 0.796 ÷ 0.85
   j. 8.32 ÷ 9.08

7. Find the following quotients to the thousandths place:

   a. 87 ÷ 17
   b. 9.62 ÷ 85
   c. 39 ÷ 7.2
   d. 48.6 ÷ 0.79
   e. 79.62 ÷ 1.5
   f. 0.83 ÷ 42
   g. 7.6 ÷ 5.23
   h. 0.897 ÷ 0.84
   i. 8.29 ÷ 1.76
   j. 6.5 ÷ 0.32

8. Harry weighed 92.5 pounds when he entered the seventh grade. Ten months later he weighed 103.4 pounds. How much did he gain, on the average, each month during this time?

9. At the start of a trip the speedometer of a car read 13243.7. At the end of the trip it read 13536.2. If 15 gallons of gasoline were used on the trip, how many miles did the car travel per gallon?

10. Mr. Jones paid $54.24 for his phone bills last year. How much did he pay, on the average, per month?

11. How many rods are there in 495 feet? (There are 16.5 feet in a rod.)

12. Eight girls shared the cost of a party. If the total expenses were $27.60, how much was each girl's share?

9-3. Repeating Decimals.

We can change any rational number expressed as a fraction to a decimal representation with as many decimal places as we please. Some of the representations are exact; others are approximate. We know that the more decimal places in an approximate representation, the better the approximation. But is it possible to find exact decimal representations for all rational numbers?

If a fraction has a denominator that is a power of ten and a numerator that is a counting number, it is easy to make the conversion. For example,

$$\frac{7}{10} = .7$$

$$\frac{23}{100} = .23$$

$$\frac{37}{1000} = .037$$

If the denominator is not a power of ten, the fraction can often be changed to an equivalent one whose denominator is a power of ten, and whose numerator is a counting number.

For example,

$$\frac{3}{20} = \frac{15}{100} = .15$$

$$\frac{8}{25} = \frac{32}{100} = .32$$

We can write a fraction in decimal form by dividing numerator by denominator and carry out the process to any desired number of places. For example, we express $\frac{3}{16}$ to four decimal places as follows:

```
     .1875
  16)3.0000
     1 6
     1 40
     1 28
       120
       112
         80
         80
```

$$\frac{3}{16} = .1875$$

We are also able to approximate fractions such as

$$\frac{1}{3}, \quad \frac{1}{6}, \quad \frac{4}{11}, \quad \frac{7}{33}, \quad \frac{23}{37}$$

as decimals, correct to any desired number of decimal places. In none of these cases do we obtain an exact representation of the original rational number. For example,

$$\frac{1}{3} \approx .333333$$

$$\frac{1}{6} \approx .166666$$

$$\frac{4}{11} \approx .363636$$

$$\frac{7}{33} \approx .212121$$

$$\frac{23}{37} \approx .621621$$

You may have noticed that in each case there seems to be a block of one, two, or three digits which repeat endlessly. This is also true of the first group of fractions in this section, if we consider the digit 0 as repeating endlessly. For example, we express these fractions as decimals to the millionths place:

$$\frac{7}{10} = .700000$$

$$\frac{23}{100} = .230000$$

$$\frac{37}{1000} = .037000$$

$$\frac{3}{20} = .150000$$

$$\frac{8}{25} = .320000$$

$$\frac{3}{16} = .187500$$

A difference between the two groups of fractions is that in one we make use of the equal sign whereas in the other we do not. (In the first group we used the sign which means "is approximately equal to.") However, here is a way to represent any rational number exactly where there is a repetition of a block of digits:

$$\frac{1}{3} \approx .333333 \qquad \frac{1}{3} = .3\overline{3}$$

$$\frac{4}{11} \approx .363636 \qquad \frac{4}{11} = .36\overline{36}$$

$$\frac{23}{37} \approx .621621 \qquad \frac{23}{37} = .621\overline{621}$$

In the first case the bar means that the digit 3 repeats endlessly; in the second one the block of digits 36 repeats; and in the third case the block 621 repeats. Note that we write the block of digits twice, and place a bar over the second block. You may also see this notation used in one of the following forms:

$\frac{1}{3} = .3\overline{3}\ldots$ or $\frac{1}{3} = .\overline{3}$ or $\frac{1}{3} = .33\overline{3}$

$\frac{4}{11} = .36\overline{36}\ldots$ or $\frac{4}{11} = .\overline{36}$ or $\frac{4}{11} = .3636\overline{36}$

When this type of repetition occurs, we have a way of naming a rational number by a decimal numeral. Let us see if the decimal form of every rational number has this repetition.

Exercises 9-3a
(Class Discussion)

1. Express $\frac{1}{7}$ as a decimal to the thousandths place. Is there a pattern of repeating digits?

$$\begin{array}{r} .142 \\ 7\overline{)1.000} \\ \underline{7} \\ 30 \\ \underline{28} \\ 20 \\ \underline{14} \\ 6 \end{array} \qquad \frac{1}{7} \approx .143$$

There is no repetition of digits yet. Let us try the ten-thousandths place. Complete the following division:

$$\begin{array}{r} .142? \\ 7\overline{)1.0000} \\ \underline{7} \\ 30 \\ \underline{28} \\ 20 \\ \underline{14} \\ 60 \end{array}$$

(Note that there was no need to repeat all that was done in the first division problem.)

2. There is still no repetition of digits. Continue the process described in Problem 1 until you find a repetition.

3. A decimal such as $.3\overline{3}$ or $.621\overline{621}$ is called a repeating decimal. Give some other examples of repeating decimals.

4. Just enough work was done below to find out whether there are repeating decimals for $\frac{16}{33}$ and $\frac{92}{111}$.

$$\begin{array}{r} .48 \\ 33\overline{)16.00} \\ \underline{13\ 2} \\ 2\ 80 \\ \underline{2\ 64} \\ 16 \end{array} \qquad\qquad \begin{array}{r} .828 \\ 111\overline{)92.000} \\ \underline{88\ 8} \\ 3\ 20 \\ \underline{2\ 22} \\ 980 \\ \underline{888} \\ 92 \end{array}$$

$\frac{16}{33} = .48\overline{48}$ $\qquad$ $\frac{92}{111} = .828\overline{828}$

How do we know that the decimal repeats after doing only this much work?

Exercises 9-3b

1. Write each of the following rational numbers as repeating decimals.

a. $\frac{1}{9}$ $\qquad$ d. $\frac{25}{37}$

b. $\frac{23}{45}$ $\qquad$ e. $\frac{5}{6}$

c. $\frac{7}{15}$ $\qquad$ f. $\frac{32}{111}$

2. Express the following rational numbers as decimals to the thousandths place.

a. $\frac{1}{11}$ $\qquad$ d. $\frac{9}{11}$

b. $\frac{2}{11}$ $\qquad$ e. $\frac{14}{11}$

c. $\frac{3}{11}$ $\qquad$ f. $\frac{23}{11}$

3. Study these decimal numerals and see if you can find a relationship:
   - a. Between the decimal numeral for $\frac{1}{11}$ and the decimal numeral for $\frac{2}{11}$.
   - b. Between the decimal numeral for $\frac{1}{11}$ and the decimal numerals for $\frac{3}{11}$, $\frac{9}{11}$, $\frac{14}{11}$, $\frac{23}{11}$.
4. Can you find a decimal numeral for $\frac{5}{11}$ without dividing?
5. Can you find a decimal numeral for $\frac{7}{11}$ without dividing?
6. Is it true that the decimal numeral for $\frac{3}{11}$ is three times that of $\frac{1}{11}$?

If you worked all the exercises correctly you will probably say that all rational numbers can be given a decimal name. However, we certainly have not tried all the rational numbers. In fact it is impossible to try all possible rational numbers. Nevertheless, we can show that every rational number can be expressed as a repeating decimal. (Remember, we agreed to call decimals like .35 repeating: .35 = .35000...)

Exercises 9-3c
(Class Discussion)

Let us look at the decimal which names the rational number $\frac{1}{7}$.

$$7\overline{)1.000000000000}\quad 0.142857142857\ldots$$

1. Can you tell, without performing the division, the digits that will appear in the next 6 places?

2. Is there a block of digits which repeats endlessly? Let us place a horizontal bar over the block of digits which repeats:

$$\frac{1}{7} = .142857\overline{142857}$$

3. Name $\frac{1}{11}$ by a decimal numeral.

4. Will there be a zero remainder if you keep on dividing?

5. Does the decimal numeral for $\frac{1}{11}$ repeat? How will you indicate this?

6. Do you notice that the decimal numeral for $\frac{1}{11}$ repeats as the remainder repeats?

7. Let us check the idea in Problem 6 by finding the decimal numeral for $\frac{1}{7}$.

$$
\begin{array}{rl}
0.1428571 & \\
7\overline{)1.0000000} & \\
\underline{7} & \\
30 & \longrightarrow \text{First remainder is } 3 \\
\underline{28} & \\
20 & \longrightarrow \text{Second remainder is } 2 \\
\underline{14} & \\
60 & \longrightarrow \text{Third remainder is } 6 \\
\underline{56} & \\
40 & \longrightarrow \text{Fourth remainder is } 4 \\
\underline{35} & \\
50 & \longrightarrow \text{Fifth remainder is } 5 \\
\underline{49} & \\
10 & \longrightarrow \text{Sixth remainder is } 1 \\
\underline{7} & \\
3 & \longrightarrow \text{Seventh remainder is } 3
\end{array}
$$

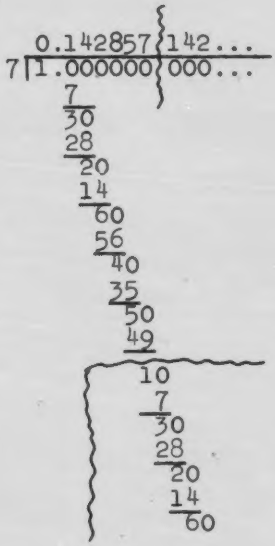
Note that the seventh remainder is the same as the first.

a. Is the seventh digit to the right of the decimal point the same as the first?

b. If you continue the division, will the eighth remainder be the same as the second remainder?

c. Will the eighth digit to the right of the decimal point be the same as the second digit to the right of the decimal point?

You should see that the block of digits, 142857, will continue to repeat. This is so because there are only zeros in the dividend. Therefore, as soon as we get to the sixth remainder and "bring down" a zero, we are back to the original problem of dividing 7 into 10.



8. Check this idea again by dividing 1 by 37 to find a decimal numeral for $\frac{1}{37}$.

You should agree with the following conclusion: Every rational number can be named by a decimal numeral which either repeats a single digit or a block of digits over and over again.

Exercises 9-3d

1. Write a decimal numeral for $\frac{1}{13}$.
   - a. How soon can you recognize a pattern?
   - b. Does this decimal numeral end?
   - c. How can you indicate that it does not end?

2. Write decimal numerals for:

   a. $\frac{1}{3}$

   b. $\frac{1}{6}$

   c. $\frac{1}{9}$

   d. $\frac{4}{11}$

   e. $\frac{7}{15}$

   How soon can you recognize a pattern in each case? In each case locate the step in which the remainders begin to repeat. Do the digits in the decimal numeral begin to repeat at this step?

3. For each of the following numbers:

   Find a decimal numeral.

   Indicate the repetition of a digit or a set of digits by placing a horizontal bar over the digit or block of digits which repeats.

   a. $\frac{1}{2}$

   b. $\frac{1}{5}$

   c. $\frac{3}{4}$

   d. $\frac{3}{5}$

   e. $\frac{7}{8}$

   f. $\frac{4}{5}$

   g. $\frac{2}{3}$

   h. $\frac{9}{11}$

   i. $\frac{5}{9}$

   j. $\frac{11}{15}$

*4. Make the following chart and fill in the blanks according to these directions:

a. In column A give the equivalent fraction with denominator 10. (Thus $\frac{3}{5} = \frac{6}{10}$.)

b. In column B give the decimal representation to the tenths place. (Thus $\frac{3}{2} = 1.5$; $\frac{5}{9} \approx .6$) Draw a circle around those decimals which are exact representations of the rational numbers under consideration (Thus there is a circle around 1.5)

c. In column C give the equivalent fraction with denominator 100.(Thus $\frac{36}{25} = \frac{144}{100}$)

d. In column D give the decimal representation to the hundredths place. (Thus $\frac{3}{4} = .75$; $\frac{1}{6} \approx .17$) Draw circles in accordance with the directions in (b). (Thus there is a circle around .75)

e. In column E give the equivalent fraction with denominator 1000.

f. In column F give the decimal representation to the thousandths place. Draw circles in accordance with the directions in (b).

	A	B	C	D	E	F
a. $\frac{3}{5}$	$\frac{6}{10}$					
b. $\frac{5}{9}$		.6				
c. $\frac{3}{2}$		1.5				
d. $\frac{36}{25}$			$\frac{144}{100}$			
e. $\frac{1}{8}$						
f. $\frac{1}{6}$				.17		
g. $\frac{3}{4}$				.75		

9-4. Rounding Decimal Numbers.

The length of the Missouri River is 2,466 miles. In talking about the length of the Missouri River a boy said that its length is 2,500 miles. Would you say that he gave the length of the river incorrectly? Actually, the boy gave the length of the river to the nearest hundred miles. We say that the length of the river was rounded to the nearest hundred miles.

We often use rounded numbers because they are easier to remember and easier to work with. In many cases, the rounded number is just as useful as the precise number.

In a recent year the population of Colorado was 1,625,089. This may be rounded to 1,600,000. We have rounded to the nearest hundred thousand in this case. In talking about populations of states people find that rounded numbers are more convenient than the exact numbers for most uses.

The height of Mt. Rainier is 14,410 feet. This may be rounded to 14,400 feet. Here the number is rounded to the nearest hundred feet.

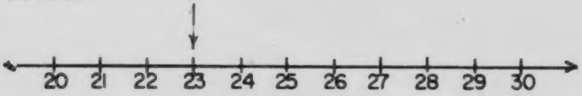
When we round a number we give an approximation of it. Often, an approximation is all we need. We may prefer the approximation because it is easier to remember and to use. As you would expect, we choose an approximation close to the actual number.

These ideas are further explained in the examples which follow in this section.

Example 1:

We say that 23, rounded to the nearest 10 units, is 20. Why don't we say 30?

Look at the number line and find 23. Is it closer to 20 or to 30? Locate the point halfway between 20 and 30. On which side of this halfway point is 23?



Example 2:

Round 677 to the nearest hundred.

Use your pencil to touch the point where you think 677 is located. What is the number for the point halfway between 600 and 700? On which side of this halfway point is 677?

Did you decide that 677, rounded to the nearest 100, is 700?

Example 3:

Round 450 to the nearest hundred.

In this case 450 is exactly halfway between 400 and 500. In such a case we could flip a coin to decide whether to round 450 to 400 or 500. In this book, however, whenever a number falls midway between two others we shall agree to round always to the larger of the two numbers. Thus 450, rounded to the nearest hundred, is 500 by agreement.

Exercises 9-4a

1. Round these numbers to the nearest hundred:

   a. 280
   b. 314
   c. 1,436
   d. 850
   e. 749
   f. 3,555
   g. 439
   h. 978
   i. 5,250

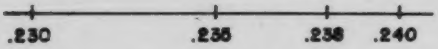
2. Round these numbers to the nearest thousand:

   a. 5,820
   b. 3,399
   c. 2,949
   d. 14,550
   e. 62,853
   f. 144,499
   g. 144,501
   h. 144,500
   i. 326,495

3. Round these numbers to the nearest ten thousand:

   a. 734,159
   b. 159,023
   c. 72,400
   d. 195,682
   e. 568,431
   f. 85,000

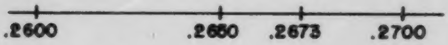
Look at the portion of the number line at the right. What number is halfway between .230 and .240? Do you agree that it is .235? Where is .238? Is it closer to .230 or to .240? Since it is closer to .240, we round .238, to the nearest hundredth, to .24.



Example 4:

Round .2673 to two decimal places.

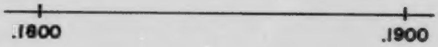
We locate .2673 between .2600 and .2700. The number halfway between .2600 and .2700 is .2650; .2673 is closer to .2700. Do you see that .2673, rounded to two decimal places, is .27?



Example 5:

Round .1839 to two decimal places.

Touch with your pencil the place where you think .1839 is on the number line. What is halfway between .1800 and .1900? On which side of the halfway point do you place .1839? Would you round off .1839 to .1800 or to .1900?



Exercises 9-4b

1. Explain why each of the following is correct:

	Number	Rounded to Two Decimal Places
a.	0.237	0.24
b.	0.241	0.24
c.	0.244	0.24
d.	0.1949	0.19

2. Round each of the following to two decimal places:

   a. 48.362
   b. 0.518
   c. 35.016
   d. 6.012
   e. 0.0053
   f. 0.097

3. Round each of the following to one decimal place:

   a. 16.38
   b. 48.72
   c. 108.05
   d. 0.074
   e. 0.051
   f. 1.16
   g. 3.1415
   h. 68.07
   i. 43.03

4. Round each of the following to three decimal places:

   a. 4.0486
   b. 17.1074
   c. 0.0006
   d. 0.00159
   e. 185.7308
   f. 62.9119

5. Write each of the following as a decimal. Round the result to one decimal place.

   a. $\frac{62}{100}$
   b. $\frac{1}{3}$
   c. $\frac{4}{9}$
   d. $\frac{5}{6}$
   e. $2\frac{1}{11}$
   f. $3\frac{4}{15}$

6. Write each of the following as a decimal. Round the result to two decimal places:

   a. $\frac{531}{1000}$
   b. $\frac{2}{3}$
   c. $\frac{5}{9}$
   d. $1\frac{7}{12}$
   e. $\frac{5}{7}$
   f. $\frac{1}{6}$

9-5. Summary.

A number may be written in decimal notation or in expanded form. For example:

Decimal Notation	Expanded Form
489.2	$4(10^2)+8(10)+9(1)+2(\frac{1}{10})$
3,056.37	$3(10^3)+0(10^2)+5(10)+6(1)+3(\frac{1}{10})+7(\frac{1}{100})$

When one reads a number, the decimal point is read "and"; such as "four hundred eighty-nine and two tenths".

The following are the procedures for performing arithmetic operations with decimals:

1. In the addition or subtraction of numbers containing decimals, write the numerals in vertical order so that the decimal points are in a column. Then add or subtract as with whole numbers. The decimal point in the answer is directly under the other decimal points.
2. The number of decimal places in the product, when two numbers are multiplied, is the sum of the number of decimal places in the two factors.
3. When dividing one decimal by another, change the problem by multiplying dividend and divisor by a suitable power of 10 so as to make the divisor a whole number. Then divide as with whole numbers. The decimal point in the answer is directly above the decimal point in the dividend.

A rational number, represented as a fraction, may be expressed as a decimal by dividing the numerator by the denominator. If the division is continued we discover that either we reach a remainder of zero or that, after a certain point, the remainders begin to repeat. After we reach a remainder of zero the succeeding digits in the quotient are zero. After we reach the first repeating remainder, the succeeding digits in the quotient are repetitions of the preceding digits in the quotient, in the same order. If we continue the division, the digits in the quotient will repeat in blocks over and over again.

Thus, each rational number may be named by a repeating decimal. In some cases, the repeating digits will be zeros. In other cases, the digits will repeat in blocks which are not all zeros.

Repeating decimals are indicated by a bar over the digits which repeat. For example, $\frac{5}{11} = .4545\ldots$ is written as $.45\overline{45}$.

In rounding decimal numerals use the following procedure:

Note the digit which is one place to the right of the place to which you are rounding. If this digit is 5 or more, then increase the preceding digit by 1. If this digit is less than 5, drop it.

9-6. Chapter Review.

Exercises 9-6

1. Write each of the following in expanded form:

a. 79.3

b. 453.08

c. 6,028.357

d. 5,739.205

2. Write each of the following in positional notation:

a. $3(10^2) + 5(10) + 2(1) + 7(\frac{1}{10})$

b. $4(10^3) + 9(10^2) + 3(10) + 7(1) + 3(\frac{1}{10}) + 8(\frac{1}{10^2})$

c. $8(10^3) + 9(10) + 4(1) + 1(\frac{1}{10}) + 0(\frac{1}{10^2}) + 9(\frac{1}{10^3})$

3. Write each of the following in words:

a. 54.3

b. 169.05

4. Write in decimal form:

   a. Six and nine tenths

   b. Ninety and six hundredths

5. Write each of the following as decimals to the nearest hundredth:

   a. $\frac{3}{10}$

   b. $\frac{5}{6}$

   c. $1\frac{3}{8}$

   d. $\frac{2}{9}$

   e. $2\frac{3}{4}$

   f. $\frac{2}{3}$

6. Change each number of the pair to a decimal rounded to the hundredths place and write them so that the larger number is first.

   a. $\frac{11}{25}$, $\frac{7}{15}$

   b. $\frac{4}{11}$, $\frac{7}{20}$

   c. $\frac{17}{53}$, $\frac{5}{18}$

7. Add the following numbers.

   a. 0.58 + 2.97

   b. 1.602 + 3.059

8. Subtract the following numbers.

   a. 0.93 - 0.48

   b. 3.057 - 1.928

9. Perform the indicated operations.

   a. (3.09 + 4.16) - 2.07

   b. (7.821 - 2.056) - 1.309

10. Find the following products.

    a. .35 × .09

    b. .008 × .7

11. Find the following quotients.

    a. .05691 ÷ .3

    b. 1.2586 ÷ .007

12. Write decimal numerals for the following:

    a. $\frac{3}{11}$

    b. $\frac{7}{9}$

    c. $\frac{5}{13}$

    How soon can you recognize a pattern in each case?

    In each case, locate the step in which the remainders begin to repeat. Do the digits in the decimal numeral begin to repeat at this step?

13. Round these numbers to the nearest hundred:

    a. 375

    b. 1250

    c. 938

    d. 12,549

14. Round these numbers to the nearest thousand:

    a. 6,712

    b. 485,350

    c. 76,802

    d. 5,349

15. Round each of the following to one decimal place:

    a. 2.47

    b. 385.063

    c. 96.55

    d. 1,043.142

16. Round each of the following to two decimal places:

    a. 3.267

    b. 489.059

    c. 0.0172

    d. 5,829.135

17. Round each of the following to three decimal places:

    a. 25.0947 c. 47.69542

    b. 3.125639 d. 78.04375

18. Write each of the following as a decimal. Round the result to two decimal places:

    a. $\frac{7}{8}$

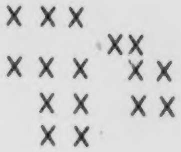
    b. $\frac{4}{17}$

9-7. Cumulative Review.

Exercises 9-7

1. What property is illustrated below?

$$5 \cdot 3 + 5 \cdot 2 = 5(3 + 2)$$



2. The number of x's at the right is written in numerals in four different bases. Which numerals are correct?

    a. 100_{four} c. 16_{ten}

    b. 14_{twelve} d. 31_{five}

3. Write 7^4 without using exponents. What is its value?

4. Answer "true" or "false".

    a. Every number can be completely factored in only one way except for the order of the factors.

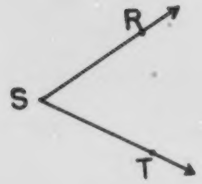
    b. Zero is a factor of every number.

    c. The number one is a factor of every number.

    d. The intersection of the set of prime numbers and the set of even numbers is the empty set.

    e. If a, b, c, and d are whole numbers and b and d are not equal to zero, then $\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$.

5. In the diagram at the right:

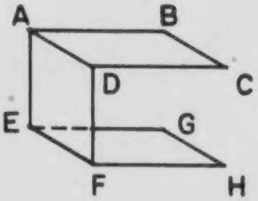


a. What is $\overrightarrow{SR} \cup \overrightarrow{ST}$?

b. What is point S called?

c. What is T-side
$\overleftrightarrow{SR} \cap$ R-side $\overleftrightarrow{ST}$?

6. In the figure on the right:



a. What is plane
ABC $\cap$ plane EFD?

b. Name two skew lines.

c. What is $\overline{AD} \cap \overline{BC}$?

d. What is $\overrightarrow{FE} \cup \overrightarrow{FH}$?

7. Perform the indicated operations and simplify.

a. $1\frac{3}{5} \times 2\frac{1}{2}$

b. $\frac{4}{5} \times 4\frac{1}{2}$

c. $1\frac{2}{3} \times 2\frac{1}{10}$

d. $4\frac{1}{2} + 4\frac{1}{5}$

e. $3\frac{3}{16} - 1\frac{3}{8}$

f. $\frac{1}{6} \div \frac{1}{12}$

g. $2\frac{1}{2} \div 1\frac{1}{4}$

h. $4\frac{1}{3} - 2\frac{1}{2}$

i. $3\frac{1}{8} + 5\frac{3}{4}$

8. Write each of the following as decimals.

a. $\frac{17}{8}$

b. $\frac{21}{4}$

c. $\frac{9}{10}$

d. $\frac{49}{56}$

e. $1\frac{3}{4}$

9. Mr. Miller is a salesman for the General Products Corporation. He is permitted an allowance of 8.5 cents per mile for car travel. What is his allowance for a trip of 634 miles?

10. Divide, and then use the inverse operation for checking.

$$28\,\overline{)84868}$$

11. By what number can you replace x, so that the statement will be true?

a. $2x = 1$

b. $7x = 14$

c. $11x = 0$

d. $3x = 4$

e. $7x = 5$

12. Insert one of the three symbols <, >, or = between the following pairs of numbers so as to make true statements.

a. $\frac{8}{12}\ ?\ \frac{2}{3}$

b. $\frac{3}{4}\ ?\ \frac{2}{3}$

c. $\frac{8}{12}\ ?\ \frac{16}{24}$

d. $\frac{7}{8}\ ?\ \frac{8}{9}$

e. $\frac{6}{7}\ ?\ \frac{11}{13}$

Chapter 10

RATIO AND PERCENT

10-1. Ratio.

In your use of numbers, you probably noticed that you can compare two numbers by subtraction or by division. Of the two numbers, 6 and 2, we can say that the first number is 4 more than the second, or that the first number is three times the second. We can also say that the ratio of the first number to the second is 3 to 1. Ratio is a comparison by division. In comparing the numbers 9 and 2, we say that the ratio of these numbers is 9 to 2 or $\frac{9}{2}$.

In a class there are 36 pupils of whom 16 are girls and 20 are boys. The ratio of the number of girls to the number of boys is $\frac{16}{20}$ or $\frac{4}{5}$. You will often find this shortened to "the ratio of girls to boys is $\frac{16}{20}$ or $\frac{4}{5}$."

When a ratio is expressed as a fraction care must be taken to write the numerals in the proper places. Thus, in the preceding example, the ratio of the number of boys to the number of girls is $\frac{20}{16}$ or $\frac{5}{4}$. This is not the same as the ratio $\frac{4}{5}$.

In general, the ratio of c to d is $\frac{c}{d}$ where the first number is used as the numerator and the second number is used as the denominator. How would you write the ratio of d to c?

Definition: The ratio of a number c to a number d, $(d \neq 0)$, is the quotient $\frac{c}{d}$. (Sometimes this is written $c : d$.)

Suppose that John is 60 inches high and that his father is 72 inches high. Then the ratio of the number of inches in John's height to the number of inches in his father's height is $\frac{60}{72}$ or $\frac{5}{6}$. Such a statement as this is usually shortened to: "the ratio of John's height to his father's height is $\frac{5}{6}$."

Notice that this statement does not tell what units are used in measuring the heights. If these heights are measured in feet, then John's height is 5 feet and his father's height is 6 feet. The ratio of the number of feet in John's height to the number of feet in his father's height is still $\frac{5}{6}$. We see that the ratio is the same no matter what units we use as long as we use the same units for both measurements.

We could say that the ratio of John's height in inches to his father's height in feet is $\frac{60}{6}$ or 10. This statement is correct and meaningful but not very useful. If, in this statement, we left out the units and said that the ratio of John's height to his father's height is 10, then people would probably make the incorrect conclusion that John is 10 times as high as his father.

When we use ratios to compare two heights, two distances, two volumes or any two measured quantities we must either use the same units for both measurements or we must clearly state the units used for the two measurements.

You may sometimes wish to compare measured quantities, lengths for example, using different units. On a map of the United States you might see the expression "1 inch = 200 miles." This means that if the distance between two points on the map is 1 inch, then the distance between the corresponding points on the ground is 200 miles.

The ratio of these lengths, 1 inch to 200 miles, is found by first expressing 200 miles in inches. Since there are 12 inches in a foot and 5280 feet in a mile, then the number of inches in 200 miles is $200 \cdot 5280 \cdot 12$ or 12,672,000. The desired ratio is then $\frac{1}{12,672,000}$. This is called the scale of the map.

Usually, in reading a map, we are not particularly interested in the scale. We could use this scale to convert inches on the map to inches on the ground. But we are more interested in converting inches on the map to miles on the ground. For this purpose we use the ratio $\frac{1}{200}$. This is the ratio of the number of inches between points on the map to the number of miles between corresponding points on the ground. This ratio will remain the same for any pair of points on the map.

Let us consider another example. Suppose a club consisted of 6 members and that one member of the group suggested that they all join an art class. After much discussion a vote was taken and the other 5 members of the group voted against the motion. We can say that the ratio of the number of members who voted against the motion to the number who voted in favor of the motion was 5 to 1. This can be written as $\frac{5}{1}$ or 5.

Exercises 10-1a

(Class Discussion)

1. Express the ratios of the following as fractions in simplest form.
    a. The number of inches in a foot to the number of inches in a yard.
    b. The number of pints in a quart to the number of pints in a gallon.
    c. The number of cents in a dollar to the number of cents in a quarter.
2. In a class there are 32 pupils of whom 20 are girls.
    a. The ratio of the number of girls in the class to the total number of class members, in simplest form, is __?__ .
    b. What fractional part of the number of class members is the number of girls?
    c. What fractional part of the number of class members is the number of boys?
    d. What is the ratio of the number of boys to the number of girls in the class?

3. On a certain map you read "1 inch = 100 miles".
   a. The distance between Hotberg and Coldspot on the map is represented by a line segment of $\frac{3}{4}$ in. How many miles are there between the cities?
   b. The airplane distance between Hotberg and Middletown is 250 miles. How far apart are the cities on the map?

We also use the idea of ratio to compare numbers which represent very different quantities.

If a car travels 258 miles in 6 hours, the ratio of the number of miles traveled to the number of hours of travel is $\frac{258}{6}$ or $\frac{43}{1}$ or 43. This ratio, 43, is usually called the rate that the car is traveling, and is often expressed in terms of miles per hour. In this example, the rate of speed is 43 miles per hour. In examples of motion, like that of a moving car, you may have used a formula

$$d = rt,$$

where d represents the number of units in the distance traveled; r, the rate of travel; t, the time of travel. If $d = 258$ and $t = 6$, then

$$6r = 258,$$

$$r = \frac{258}{6} = 43.$$

Exercises 10-1b

1. Find the ratio of Helen's height to Laurie's height. Helen is 48 inches tall, and Laurie is 64 inches tall.
2. Josephine has 90 cents and Martin has 30 cents. What is the ratio of Josephine's amount of money to Martin's amount?
3. Marc walked 5 miles and Jerry walked 3 miles. What is the ratio of the distance Jerry walked to the distance Marc walked?

4. What is the ratio of Marc's distance to Jerry's distance in Problem 3?

5. Bus fare ten years ago was 10 cents. Now it is 30 cents. Find the ratio of the present fare to the former fare.

6. How much did the fare in Problem 5 increase? Find the ratio of the increase to the former fare.

7. During a sale, \$8.00 shoes were sold for \$6.00. Find the ratio of the decrease in the cost to the original price.

8. In Problem 7, find the ratio of the sale price to the original price.

9. The temperature rose from 80 degrees to 90 degrees. Find the ratio of the increase to the original temperature.

10. Express the ratios of the following as fractions in simplest form.
    a. $\frac{9}{2}$ to $\frac{7}{2}$
    b. $\frac{5}{4}$ to 2
    c. $\frac{3}{4}$ to $\frac{5}{2}$
    d. $2\frac{1}{2}$ to $1\frac{1}{4}$

11. The scale on a map is 1 inch to 20 miles.
    a. Express this scale as a ratio.
    b. On the map how many miles are represented by a segment of length $4\frac{1}{4}$ inches?

12. Centerville is a small city with city limits forming the sides of a rectangle, 3 miles in length on the longer side, and 2 miles in length on the shorter side. Using a scale of 1 inch for $\frac{1}{8}$ of a mile, how long and how wide will the map of the city have to be?

13. A map of the United States is drawn on a scale of 1 inch = 300 miles.
    a. Express this scale as a ratio.
    b. On this map how many miles will be represented by one foot?
    c. The distance from Washington to Chicago is approximately 750 miles. How far apart will these cities be on the map?

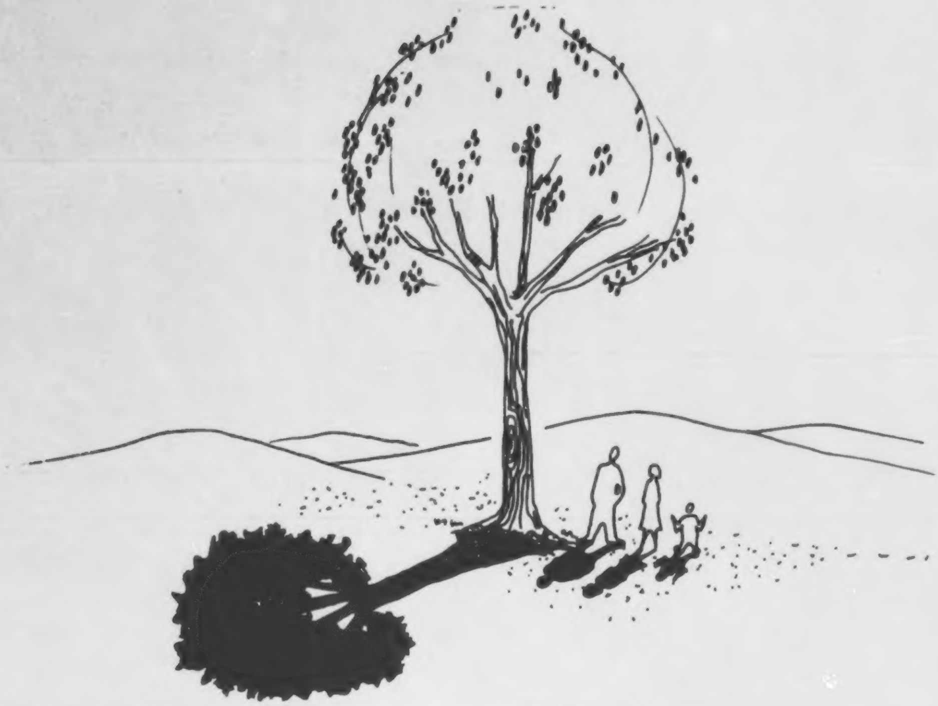
14. A plane flies 2600 miles in 5 hours.
    a. At what rate does the plane fly per hour?
    b. What is the ratio of the number of miles traveled to the number of hours of flying time?

15. A balloon drifts 600 miles in 15 hours. What is the rate at which the balloon drifts per hour?

One sunny day Tony measured the length of the shadow made by each member of his family. He also measured the length of the shadow made by a big tree in the yard. He found that his father, who is 72 inches tall, had a shadow 48 inches long. His mother, who is 63 inches tall, had a shadow 42 inches long. His little brother, who is only 30 inches high, had a shadow 20 inches long. He didn't know how tall the tree was but its shadow was 40 feet long.

He wrote this information in a table.

	Shadow length	Height
Father	48 inches	72 inches
Mother	42 inches	63 inches
Brother	20 inches	30 inches
Tree	40 feet	?



Tony saw that the taller people had longer shadows. He wondered if he could find a useful way to compare the numbers, and tried division. He divided the shadow length of his little brother by the number of inches in his brother's height.

$$\frac{20}{30} = \frac{2 \cdot 10}{3 \cdot 10} = \frac{2}{3}$$

He tried the same thing with his father's shadow length and height. He formed the ratio of shadow length to the height of his father.

$$\frac{48}{72} = \frac{2 \cdot 24}{3 \cdot 24} = \frac{2}{3}$$

What is the ratio of the shadow length of his mother to her height?

All heights and shadows were measured at the same time. The ratio $\frac{2}{3}$ is the same for all of them. This ratio $\frac{2}{3}$ is the same for the tree, too.

The shadow of the tree was measured to be 40 feet. Can you find the height of the tree? What must the height of the tree be in order that the measure of its shadow length divided by the number of feet in its height is $\frac{2}{3}$? $40 \div ? = \frac{2}{3}$. With this reasoning you can find the height of the tree without measuring it. Did you find 60 feet?

Exercises 10-1c

1. What is your height in inches? What would be the length of your shadow if it were measured at the same time and place as the shadows of the people in our story were measured?
2. Some other objects were measured at another time and place, and the data are recorded below. Copy and complete the table.

Object	Shadow Length	Height	Ratio
Garage	3 feet	8 feet	
Clothes pole	36 inches		$\frac{3}{8}$
Tree	$7\frac{1}{2}$ feet	20 feet	
Flagpole		144 inches	$\frac{3}{8}$
Fence	$11\frac{1}{4}$ inches	30 inches	

3. Write the ratio of
   a. 2 to 9
   b. 48 to 40
   c. 175 to 125
   d. 65 to 100
4. Write an expression for the ratio of c to d.
5. Write an expression for the ratio of s to 66.
6. Write an expression for the ratio of $2k$ to $7m$.
7. Write an expression for the ratio of 40 to w.

10-2. Proportion.

Tony's older brother is 66 inches tall. Is it possible to find the length his shadow would have been when Tony made the other measurements?

To answer this question we shall write a fraction for the ratio of his older brother's shadow length to his height and state that it is the same as the other ratios. We shall use s to represent the number of inches in his shadow length which we wish to find.

$$\frac{s}{66} = \frac{2}{3}$$

Now we have two names for the same ratio. We know how to express a rational number in different ways. It will help us to express $\frac{2}{3}$ as a fraction with denominator 66.

$$\frac{2}{3} = \frac{2 \cdot 22}{3 \cdot 22} = \frac{44}{66}$$

We can use $\frac{44}{66}$ instead of $\frac{2}{3}$ and write

$$\frac{s}{66} = \frac{44}{66}$$ (The denominators are the same.)

This tells us that $s = 44$.

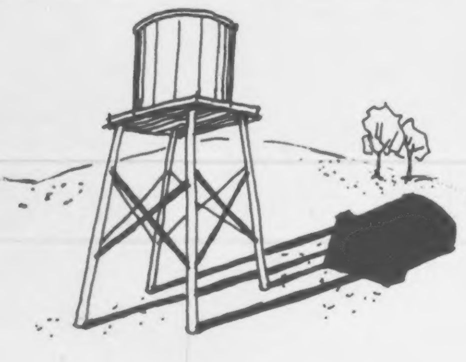
The older brother's shadow would have been 44 inches long.

The statement $\frac{s}{66} = \frac{2}{3}$ states that two ratios are equal. A statement that two ratios are equal is called a proportion.

Later in the day, the ratio of shadow length to height changed to $\frac{5}{7}$. At this time the shadow of a water tower measured 21 feet. Then the statement about the height, w, of the water tower,

$$\frac{21}{w} = \frac{5}{7}$$

is a proportion. We wish to find the height. To solve problems like this we need a method for finding a number



which will make the statement true when it is used as a replacement for w.

The statement says that the two fractions name the same rational number. From Chapter 8 we know the Comparison Property:

$$\text{If } \frac{a}{b} = \frac{c}{d} \text{ and } b \neq 0,\ d \neq 0, \text{ then } a \cdot d = b \cdot c.$$

We use this property to solve our problem

$$\frac{21}{w} = \frac{5}{7}$$

$$21 \cdot 7 = 5 \cdot w \qquad \text{Comparison Property}$$

$$5w = 147$$

$$w = \frac{147}{5} \qquad \text{Definition of a rational number}$$

$$w = 29\tfrac{2}{5} \qquad \text{The water tower is about 29 feet in height.}$$

The problem of the older brother's shadow can be solved in the same way.

$$\frac{s}{66} = \frac{2}{3}$$

$$3 \cdot s = 2 \cdot 66 \qquad \text{State the property used here.}$$

$$3s = 132$$

$$s = 44$$

The older brother had a 44-inch shadow.

Exercises 10-2

1. Find the ratio of the first number to the second.

	Shadow	Height	Ratio
a. Garage:	3 feet	10 feet	?
b. Clothes pole:	36 inches	96 inches	?
c. Fence:	11 inches	30 inches	?

2. In a mixture of nuts weighing 24 pounds there are 15 pounds of peanuts.
   a. How many pounds of other nuts are there in this mixture?
   b. What is the ratio of the number of pounds of peanuts to the total number of pounds of nuts in the mixture?
   c. What is the ratio of the number of pounds of other nuts to the total number of pounds of nuts in the mixture?
   d. What is the ratio of the number of pounds of peanuts to the number of pounds of other nuts?

3. In another mixture, the ratio of the number of pounds of peanuts to the number of pounds of other nuts is the same as in Problem 2. This mixture contains 25 pounds of peanuts. How many pounds of other nuts does it contain?

4. What number substituted for n will make the statement true?

   a. $\frac{n}{18} = \frac{4}{9}$ b. $\frac{56}{n} = \frac{7}{3}$ c. $\frac{16}{5} = \frac{96}{n}$

5. What number substituted for s will make the statement true?

   a. $\frac{20}{8} = \frac{s}{6}$ b. $\frac{30}{14} = \frac{90}{s}$ c. $\frac{s}{42} = \frac{36}{27}$

6. Joyce has a picture 4 inches wide and 5 inches long. She wants an enlargement that will be 10 inches wide. How long will the enlarged print be?

7. Mr. Stephens was paid $135 for a job he finished in 40 hours. At this rate how much should he be paid for 60 hours of work?

8. A recipe for 30 cookies listed the following amounts:

   1 cup butter
   $\frac{2}{3}$ cup sugar
   2 eggs
   $1\frac{1}{2}$ cups flour
   1 tsp. vanilla

   a. Re-write the recipe to make 90 cookies.
   b. Suppose you wish to make 45 cookies using the original recipe. What ratio should you use? What amount of each item should you take?

9. What is the cost of 10 doughnuts at 50 cents a dozen?

10. What is the cost of 12 candy bars at 3 for 25¢ ?

11. What is the cost of 2500 bricks at $14 per 1000 bricks?

12. The following table lists pairs of numbers v and w. In each pair the ratio of v to w is the number $\frac{6}{7}$. Copy and complete the table.

	v	w	ratio $\frac{v}{w}$ (simplest form)
a.	12	14	$\frac{6}{7}$
b.		21	
c.	30		
d.		100	
e.	100		

13. State which of the following ratios are equal. Use the Comparison Property as your test.

a. $\frac{10}{5}$, $\frac{20}{10}$

b. $\frac{6}{3}$, $\frac{16}{9}$

c. $\frac{48}{16}$, $\frac{42}{14}$

d. $\frac{68}{17}$, $\frac{76}{19}$

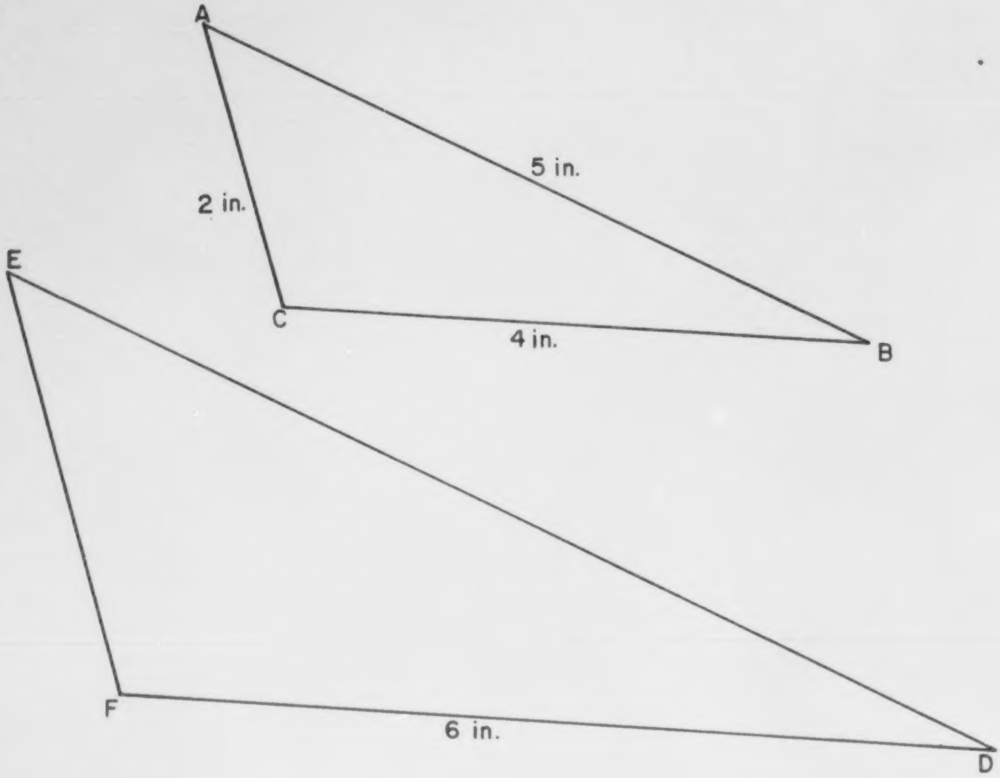
e. $\frac{15}{13}$, $\frac{45}{39}$

14. Mr. Jones is told to mix 2 pints of pigment with 3 gallons of paint to complete a certain paint job. He later finds that he wishes to mix pigment with 2 gallons of paint. How much pigment does he need?

*15. When triangles are the same shape but not the same size, they are called similar triangles. The measures of the lengths of corresponding sides of similar triangles form equal ratios. Triangles ABC and DEF are similar. By using equal ratios,

a. Find the length of side $\overline{EF}$.

b. Find the length of side $\overline{DE}$.



10-3. Percent--A Special Kind of Ratio.

Many of you are familiar with the word "percent", and you may know something about its meaning. When your teacher says "90 percent of the answers on this paper are correct," do you know what he means? Actually he is talking about a number, the ratio of the number of questions you answered correctly to the total number of questions on the test. If the paper with 90 percent of the answers correct has 100 questions, then 90 answers out of the 100 are correct. The ratio $\frac{90}{100}$ could be used instead of the phrase "90 percent" to describe the part of the questions which are answered correctly. The word "percent" is used when a ratio is expressed with a denominator of 100.

Ninety percent means the ratio of 90 to 100 or $\frac{90}{100}$.

The word "percent" means hundredths, and comes from the Latin phrase "per centum" which means "by the hundred". For convenience the symbol "%" is used for the word "percent". This symbol is a short way of saying "times $\frac{1}{100}$" or "times 0.01".

$$90\% = 90 \times \frac{1}{100} = \frac{90}{100}$$

$$\text{Or} \quad 90\% = 90 \times 0.01 = 0.90$$

You should see that the following are equivalent forms of expressing the same number:

$$90\% = \frac{90}{100} = 0.90$$

There are other forms. Thus $\frac{90}{100} = \frac{9}{10}$. The simplest fraction for 90% is $\frac{9}{10}$.

Exercises 10-3a
(Class Discussion)

1. Express as fractions in simplest form:

   7%, 13%, 20%, 25%, 37%, 50%, 75%, 100%, 150%.

2. Express as decimals:

   17%, 25%, 65%, 10%, 8%, 100%, 150%, 200%.

Exercises 10-3b

1. Using squared paper draw a large square so that the interior is divided into 100 small squares. Write the letter A in each of 10 small squares. Write the letter B in 20 of the squares. Write the letter C in 35 of the squares. Write the letter D in 30 of the squares. Write the letter X in the remainder of the squares.
   - a. What is the ratio of the number of squares which contain the letter A to the total number of squares?
   - b. In what percent of the squares is the letter A?
   - c. For each of the remaining letters B, C, D, and X, write the ratio of the number of squares which contain the letter to the total number of squares. Give each of these ratios in simplest form.
   - d. What is the sum of the ratios in Parts (a) and (c)?
   - e. In what percent of the squares is the letter B?
   - f. In what percent of the squares is the letter C?
   - g. In what percent of the squares is the letter D?
   - h. In what percent of the squares is the letter X?
   - i. What is the sum of the percents of the squares containing the letters A, B, C, D, X?

2. Copy the table below and fill in the blank spaces as shown in the first line.

a.	$\frac{1}{4}$	$\frac{25}{100}$	$25 \times \frac{1}{100}$	25%
b.	$\frac{2}{5}$	$\frac{40}{100}$	$40 \times \frac{1}{100}$	____
c.	$\frac{8}{5}$	$\frac{160}{100}$	________	____
d.	$\frac{7}{10}$	____	________	____
e.	$\frac{3}{4}$	____	________	____
f.	$\frac{12}{5}$	____	________	____

3. Copy the table below and fill in the blank spaces as shown in the first line.

a.	0.3	0.30	30 × 0.01	30%
b.	4	4.00	400 × 0.01	___
c.	0.78	0.78	___	___
d.	0.9	___	___	___
e.	8	___	___	___
f.	1.2	___	___	___

4. Florie has a weekly allowance of 50 cents. One week she spends 12 cents for a pencil, 10 cents for an ice cream cone, 15 cents for Sunday School collection, and puts the rest in her piggy bank.
   a. Express the ratio of each amount to her total allowance, and express each of these ratios as a percent.
   b. Find the sum of the ratios.
   c. Find the sum of the percents.
   d. What do you observe about the answers above?

5. The following boys and girls live on the same street.

Edward	Polly	Mary	Doug	Mike
Sara	Harry	Ann	Lars	Bill

   a. What percent of the total number of children are boys?
   b. What percent are girls?
   c. What is the sum of the two percents?

6. What is meant by each of the following?
   a. This fabric is 100% wool.
   b. We have had 100% attendance for a week.
   c. 100% of the members were in favor of a picnic.

Percent is a convenient tool for giving information involving ratios. Athletic standings often are given in percent. Two seventh grade pupils discovered the reason for this use of percent as they talked about their neighborhood baseball teams. The boys were discussing the scores of their baseball teams. In the Little League the first boy's team won 15 games out of 20 games played. The other boy's team won 18 out of 25 games. Which team had the better record? The second team won 3 more games, but the first team played fewer games. Look at the ratios of the number of games won to the number of games played for each team. The ratios $\frac{15}{20}$ and $\frac{18}{25}$ are not easy to compare at a glance. Percent makes the comparison easy.

The first team won $\frac{15}{20}$ of the games played:

$$\frac{15}{20} = \frac{75}{100} = 75\%.$$

The first team won 75% of the games played.

The second team won $\frac{18}{25}$ of the games played:

$$\frac{18}{25} = \frac{72}{100} = 72\%.$$

The second team won 72% of the games played.

The first team which won 75% of its games had a higher standing than the second team which won 72% of its games. We could say that $72\% < 75\%$, or $75\% > 72\%$.

Exercises 10-3c

1. A Little League team won 3 out of the first 5 games played.
    a. What percent of the first 5 games did the team win?
    b. What percent of the first 5 games did the team lose?
    c. What is the total of the percents in Parts (a) and (b)?

2. Later in the season the team in Problem 1 had won 8 out of 16 games played.
   a. What was the percent of games won at this time?
   b. Did the percent of games won increase or decrease?
3. At the end of the season, the team in Problem 1 had won 26 games out of 40.
   a. What percent of the games played did the team win by the end of the season?
   b. How does this percent compare with the other two?

Information from business, industry, school, Scouts, and similar sources is often given in percent. It is often more convenient to refer to the information at some later time if it is given in percent than if it is given in another form.

A few years ago the director of a Boy Scout camp kept some records for future use. Some information was given in percent, and some was not. The records gave the following items of information.

(1) There were 200 boys in camp.
(2) One hundred percent of the boys were hungry for the first dinner at camp.
(3) Fifteen percent of the boys forgot to pack a toothbrush, and needed to buy one at camp.
(4) On the second day at camp 44 boys caught fish.
(5) One boy wanted to go home the first night.
(6) A neighboring camp director said, "Forty percent of the boys at our camp will learn to swim this summer. We shall teach 32 boys to swim."

From (1) and (2), how many hungry boys came to dinner the first day?

Of course we know, without computation, that 100% of anything is all of it. Then 100% of 200 is 200. The answer is that 200 hungry boys came to dinner the first day.

From (3), how many extra toothbrushes were needed?

15% means $15 \cdot \frac{1}{100}$ or $\frac{15}{100}$.

The ratio of boys without toothbrushes to the total number of boys in camp is $\frac{?}{200}$. This can be written $\frac{n}{200}$ where n is the number of toothbrushes needed.

The ratio $\frac{n}{200}$ equals the ratio $\frac{15}{100}$ since 15% of the boys forgot to pack a toothbrush. We write the proportion and solve it to answer the question.

$$\frac{n}{200} = \frac{15}{100}$$

$$100n = 15 \cdot 200$$

$$n = \frac{15 \cdot 200}{100}$$

$$n = 30$$

30 toothbrushes were needed.

Statement (4) says that 44 boys caught fish. The percen of boys who caught fish can be found. The ratio of the 44 boys to the 200 boys in camp is $\frac{44}{200}$. If y percent caught fish, this ratio is $\frac{y}{100}$. Since the ratios are equal, then

$$\frac{44}{200} = \frac{y}{100}$$

$$44 \cdot 100 = 200y$$

$$200y = 4400$$

$$y = \frac{4400}{200}$$

$$y = 22$$

Then y percent is 22 percent or 22%.
22% of the boys caught fish on the second day.

From Statement (5) we can find the percent of the boys who were homesick. One boy wanted to go home the first night. So we form the ratio of one boy to the total number of boys, or $\frac{1}{200}$. If v percent wanted to go home, we have the ratio $\frac{v}{100}$.

These are equal, so

$$\frac{1}{200} = \frac{v}{100}$$

$$200v = 100$$

$$v = \frac{100}{200}$$

$$v = \frac{1}{2}$$

Then $\frac{1}{2}\%$ of the boys wanted to go home. This may be read, "one-half percent" or "one-half of one percent", preferably the latter.

Be sure that you know the difference between $\frac{1}{2}$ and $\frac{1}{2}\%$.

Other names for $\frac{1}{2}$ are $.5$, $\frac{5}{10}$, $\frac{50}{100}$, 50%.

Other names for $\frac{1}{2}\%$ are $\frac{\frac{1}{2}}{100}$, $\frac{1}{200}$, $\frac{.5}{100}$, $.005$, $.5\%$.

Which is the larger number, $\frac{1}{2}$ or $\frac{1}{2}\%$? Since $\frac{1}{2}\% = \frac{1}{2} \cdot \frac{1}{100}$ so that $\frac{1}{2}\% = \frac{1}{200}$ we see that $\frac{1}{2}$ is greater than $\frac{1}{2}\%$.

Statement (6) of the camp problem says that 40% of the boys in a neighboring camp would learn to swim; also, that 32 boys would learn to swim. How many boys are in the neighboring camp? The ratio of the number of new swimmers to the number of boys in their camp is $\frac{32}{n}$ where n is the number of boys in their entire camp. The ratio is also $\frac{40}{100}$. These two ratios name the same number, so

$$\frac{40}{100} = \frac{32}{n}$$

$$40n = 3200$$

$$n = \frac{3200}{40}$$

$$n = 80$$

There were 80 boys in the neighboring camp.

Now you have had examples of all the kinds of problems which will arise for you in percent.

Notice that each kind of problem, although very different, was solved by writing a proportion. We solved

$$\frac{n}{200} = \frac{15}{100}, \qquad \frac{44}{200} = \frac{y}{100}, \qquad \frac{40}{100} = \frac{32}{n}.$$

In each case we found two different names for the same ratio and solved the proportion. One of the ratios was always a fraction with denominator 100. There was always one unknown number, but it appeared in different places. Yet our method of solution of the proportion was always the same.

Exercises 10-3d
(Class Discussion)

1. Usually 8% of the pupils who buy season passes to school events fail to use them. If 250 pupils bought passes, how many pupils probably will not use them? Use these steps in solving the problem.
    a. Explain where $\frac{8}{100}$ comes from in setting up the solution of this problem.
    b. Explain where $\frac{n}{250}$ comes from, if n is the number of pupils who probably will not use their passes.
    c. Explain why $\frac{8}{100} = \frac{n}{250}$.
    d. Find the value of n which makes the statement in (c) true.
    e. Would you accept an answer of 200 as sensible? Why?

2. There were 300 boys in the school and 120 signed up for track practice. What percent signed up for track?
   a. What is the ratio of the number of boys who signed up for track to the total number of boys in the school?
   b. Call x the percent of boys who signed up for track.
   c. Write a fraction for (b) using 100 as the denominator.
   d. Write the statement that the ratios in (a) and (c) name the same number.
   e. Find the value of x which makes the statement in (d) true.
   f. Answer the question in the problem.
      Is your answer a sensible answer for the problem?

Exercises 10-3e

Problems 1 - 5 all refer to the same junior high school.

1. There are 600 seventh grade pupils in a junior high school. The principal plans to divide the pupils into 20 sections of equal size.
   a. How many pupils will be in each section?
   b. What percent of the pupils will be in each section?
   c. How many pupils are 1% of the number of pupils in the seventh grade?
   d. How many pupils are 10% of the number of pupils in the seventh grade?
2. Suppose one section contains 36 pupils. What percent of the seventh grade pupils are in that section?
3. One hundred fifty seventh grade pupils come to school on the school bus.
   a. What percent of the seventh grade pupils come by school bus?
   b. What percent of the seventh grade pupils come to school by some other means?

4. In a class of 30 pupils, 3 were tardy one day.
   a. What fractional part of the class do the tardy pupils represent?
   b. What percent of the class was tardy?

5. One day the principal said, "Four percent of the ninth graders are absent today." The list of absentees had 22 names of ninth graders on it. From these two pieces of information find the number of ninth grade pupils in the school.

6. Find the missing number n which will make each of the following statements true. First write a proportion, and then solve it by the method you have just learned.
   a. 50% of 75 is n.
   b. n percent of 48 is 12.
   c. 150 is 75% of n.
   d. 125% of n is 100.
   e. n is 29% of 240.
   f. 2 is n percent of 40.
   g. 30 is n percent of 25.

7. BRAINBUSTER. My brother had $72. This was 300% of what I had. My brother gave me $\frac{2}{3}$ of his money. Now I have 300% of my brother's money. How much money do I have now?

10-4. Ratio as a Percent, a Decimal, a Fraction.

There are four common ways of expressing the same ratio. You should be able to use any of these. They are: a fraction in simplest form, a fraction with 100 as the denominator, a decimal, and a percent. It is necessary to be skillful in writing different names for the same ratio in order to be free to choose the most useful form.

Exercises 10-4a
(Class Discussion)

If you wish to write a percent as a decimal you may follow these steps.

Example 1. $65\% = 65 \times 0.01 = 0.65$

Example 2. $12\frac{1}{2}\% = 12.5 \times 0.01 = 0.125$

1. Express the following as decimals: 17%, 2%, $3\frac{1}{2}\%$, 65%, 115%.

To write a percent as a fraction in simplest form, you may follow these steps.

Example 1. $65\% = 65 \times \frac{1}{100} = \frac{65}{100} = \frac{13}{20}$

Example 2. $12\frac{1}{2}\% = 12\frac{1}{2} \times \frac{1}{100} = \frac{25}{2} \times \frac{1}{100} = \frac{25}{200} = \frac{1}{8}$

Example 3. $125\% = 125 \times \frac{1}{100} = \frac{125}{100} = \frac{5 \cdot 25}{4 \cdot 25} = \frac{5}{4}$

2. Express the following as fractions in simplest form: 25%, 10%, 5%, 65%, 110%, 200%.

To write a decimal as a percent you may follow these steps:

Example 1. $0.23 = 23\%$ or $0.23 = \frac{23}{100} = 23\%$

Example 2. $0.05 = 5\%$ or $0.05 = \frac{5}{100} = 5\%$

Example 3. $0.472 = 47.2\%$ or $0.472 = \frac{47.2}{100} = 47.2\%$

3. Name the following as percents: .45, .045, 4.5, .425, 4.25.

To write a fraction as a percent, find an equivalent fraction with denominator 100. Its numerator is the required percent.

Example 1. $\frac{1}{3} = \frac{?}{100}$

Solve $\frac{1}{3} = \frac{n}{100}$

$$1 \times 100 = 3 \times n$$

$$n = \frac{100}{3}$$

$$n = 33\frac{1}{3}$$

$$\frac{1}{3} = \frac{33\frac{1}{3}}{100} = 33\frac{1}{3}\%$$

Example 2. What percent is $\frac{5}{13}$?

$$\frac{5}{13} = \frac{n}{100}$$

$$5 \times 100 = 13 \times n$$

$$n = \frac{500}{13} = 38\frac{6}{13}$$

$$\text{So } \frac{5}{13} = \frac{38\frac{6}{13}}{100} = 38\frac{6}{13}\%.$$

Since fractions can be written as decimals by just carrying out the indicated division, we could have done the previous examples this way.

Example 1. $\frac{1}{3} = 0.33\overline{3} = 33\frac{1}{3}\%$.

Example 2. $\frac{5}{13} \approx .385 = 38.5 \times 0.01 = 38.5\%$.

Note that it is necessary to use the "approximately equal" sign. Since the decimal for $\frac{5}{13}$ does not terminate, $\frac{5}{13}$ is not exactly the same as 0.385. The equal sign cannot be used. The two wavy lines $\approx$ are used to mean "approximately equal".

Exercises 10-4b

1. Write the following fractions in the form indicated:

   a. $\frac{3}{4} = \frac{?}{100} = \underline{\ ?\ }\ \%$

   b. $\frac{4}{5} = \frac{?}{100} = \underline{\ ?\ }\ \%$

   c. $\frac{7}{10} = \frac{?}{100} = \underline{\ ?\ }\ \%$

   d. $\frac{19}{25} = \frac{?}{100} = \underline{\ ?\ }\ \%$

   e. $\frac{11}{20} = \frac{?}{100} = \underline{\ ?\ }\ \%$

   f. $\frac{9}{25} = \frac{?}{100} = \underline{\ ?\ }\ \%$

   g. $\frac{17}{20} = \frac{?}{100} = \underline{\ ?\ }\ \%$

   h. $\frac{3}{5} = \frac{?}{100} = \underline{\ ?\ }\ \%$

2. Write the following fractions in decimal form. Round to the nearest hundredth.

   a. $\frac{1}{3}$

   b. $\frac{5}{6}$

   c. $\frac{3}{8}$

   d. $\frac{7}{9}$

   e. $\frac{6}{11}$

   f. $\frac{43}{49}$

   g. $\frac{86}{127}$

   h. $\frac{265}{591}$

3. Write the percent form of the decimals in Problem 2. Notice that when you round to the nearest hundredth in a decimal, the percent form is rounded to the nearest whole percent.

4. Write the following fractions in decimal form. Round to the nearest thousandth.

   a. $\frac{3}{8}$

   b. $\frac{2}{3}$

   c. $\frac{8}{9}$

   d. $\frac{29}{30}$

   e. $\frac{5}{6}$

   f. $\frac{58}{65}$

   g. $\frac{126}{490}$

   h. $\frac{387}{462}$

5. Write the percent form of the decimals in Problem 4. To what place do you round a decimal when you wish the percent form to be rounded to the nearest tenth of a percent?

If one of the four common ways of expressing a ratio is given, you can find the other three.

Example 1. $0.28 = \frac{28}{100} = 28\% = \frac{7}{25}$

$0.15 = ? = ?\% = ?$

Example 2. $72\% = 0.72 = \frac{72}{100} = \frac{18}{25}$

$48\% = ? = ? = ?$

Example 3. $\frac{2}{5} = \frac{40}{100} = 0.40 = 40\%$

$\frac{14}{25} = ? = ? = ?\%$

Example 4. $\frac{5}{6} = \frac{83\frac{1}{3}}{100} = 83\frac{1}{3}\% = 0.83\overline{3}$

$\frac{2}{3} = ? = ?\% = ?$

Example 5. $\frac{3}{100} = 3\% = .03$

$\frac{19}{100} = ?\% = ?$

In Example 5 we had only three common ways of expressing the ratios because "$\frac{3}{100}$" and "$\frac{19}{100}$" are at the same time fractions in simplest form and fractions with 100 as the denominator.

Exercises 10-4c

1. Fill in the missing numerals in the chart below. The completed chart will be helpful to you in future lessons.

Fraction in Simplest form		Hundred as Denominator	Decimal	Percent
a.	$\frac{1}{2}$	$\frac{50}{100}$	.50	50%
b.	$\frac{1}{4}$			
c.		$\frac{75}{100}$		
d.			.20	
e.				40%
f.		$\frac{60}{100}$		
g.	$\frac{4}{5}$			
h.			$.3\overline{3}$	
i.		$\frac{70}{100}$		
j.			$.6\overline{6}$	
k.	$\frac{3}{10}$			
l.			.10	
m.				90%
n.	$\frac{1}{8}$			
o.		$\frac{300}{100}$		
p.			.375	
q.				150%
r.		$\frac{62\frac{1}{2}}{100}$		
s.			.01	
t.	$\frac{7}{8}$			
u.				100%
v.		$\frac{16\frac{2}{3}}{100}$		
w.	$\frac{5}{6}$			

2. Draw a number line and at points about 5 inches apart mark 0 and 100%.
   a. Locate the following percents from Problem 1 on this line. Estimate their positions and mark the percent names below the line. a, b, c, d, e, f, g, h, n, p, t, w.
   b. Mark these fractions above the appropriate points on the line.

      $\frac{1}{5}, \frac{1}{3}, \frac{1}{4}, \frac{2}{5}, \frac{1}{2}, \frac{3}{4}, \frac{3}{5}, \frac{7}{8}, \frac{4}{5}, \frac{2}{3}, \frac{3}{8}, \frac{5}{6}.$
3. Using squared paper, draw 5 large squares each containing 100 small squares. By proper shading show the percents in b, d, l, p, s of Problem 1.
4. What fraction in simplest form is another numeral for
   a. 32%   b. 90%   c. 120%
5. Express the following as percents.
   a. $\frac{13}{25}$ b. $\frac{7}{20}$ c. $\frac{19}{20}$ d. $\frac{3}{10}$

10-5. Applications of Percent.

Percent is used to express ratios of numerical quantities in everyday experience. It is important for you to understand the idea of percent, the way it is written, and, also, to be accurate in working with percent.

Budgets

A budget is a plan for spending money. Families use budgets to help them work out how their money will be spent for food, housing, personal needs, savings and other things. Governments, businesses, and school systems all have budgets to show how they plan to spend their incomes. Boys and girls in school often have budgets, too.

Suppose a family had a monthly income of $410 after taxes. The family budget allowed 32% for food. How much money is 32% of $410?

The problem is to find m, the number of dollars allowed for food. The ratio $\frac{32}{100}$ is known. Tell why this is known. The ratio $\frac{m}{410}$ is another way to express the ratio of the number of dollars allowed for food to the number of dollars of income. Then

1. $\frac{m}{410} = \frac{32}{100}$ Tell why.

2. $100m = 32 \times 410$ Explain this step.

3. $m = \frac{13120}{100}$ Explain this step.

4. $m = 131.20$

$131.20 is the amount allowed for food.

Suppose the family rents an apartment for $78 per month. What percent of the family income of $410 per month is spent for rent? The problem is to find the percent of income spent for rent. Let us call this ratio $\frac{r}{100}$. The known ratio is $\frac{78}{410}$. Notice that this is about $\frac{80}{400}$ or about $\frac{1}{5}$. A good estimate then, is about 20%. It is always sensible to estimate the answer before you work the problem.

$$\frac{r}{100} = \frac{78}{410}$$

$$r = \frac{78 \times 100}{410}$$

$$r \approx 19.02$$

The percent of income spent for rent is about 19%.

The wavy lines $\approx$ mean that r is not exactly the same as 19.02 since the decimal does not terminate in the hundredths place. The value 19.02 is an approximation for r. The rounded answer 19% is close enough for practical purposes.

Exercises 10-5a

Suggested Budget for a Family of Four

Take-home pay	$300 per mo.	$350 per mo.	$450 per mo.	$550 per mo.
Food	30 %	27 %	22%	22 %
Housing	30 %	28 %	28%	28 %
Personal needs	13 %	13%	12%	12%
Transportation	7%	7%	9%	9%
Savings, Taxes	5%	10%	16%	16%
Others	15%	15%	13%	13%
Total	100%	100 %	100%	100 %

1. Why should each column total 100%?

2. What percent of the $450 budget is allowed for food?

3. a. Write the ratio in Problem 2 as a fraction with 100 as the denominator.
   b. Use f to stand for the number of dollars allowed for food, and write the ratio of f to $450.
   c. 22% is about $\frac{1}{5}$. Use this to estimate the amount of money spent on food out of a $450 income.
   d. Use the ratios in (a) and (b) to write a proportion and find the number, f.
   e. Check your answer with the estimate in (c).

4. a. Write the percent of the $550 income which is allowed for the item called "Savings, Taxes".
   b. Find the number of dollars of the $550 income which are allowed for Savings, Taxes.
   c. Notice that 16% is between 10% and 20%. Then it is between $\frac{1}{10}$ and $\frac{1}{5}$. How can you use this information to check your answer?

5. What amount, according to the table, is allowed for personal needs from an income of $300?

6. Find the amount allowed for transportation when the income is $350.

7. Find the amount allowed for housing when the income is $450 a month.

8. Find the amount allowed for savings and taxes when the income is $300 per month.

9. The Hughes family of 4 has a monthly income of $575 after taxes. The amount this family spends for housing averages $150 a month. Find the percent of their income that they use for housing, to the nearest whole percent. Compare it with the percent allowed in the table for an income of $550.

*10. An apartment for a family of 5 rents for $95 a month. If the family plans to spend 25% of the monthly income for rent, what monthly income does the family need?

Commissions and Discounts

People who work as salesmen often are paid a commission instead of a salary. The money they earn depends upon the sales they make. A salesman, for example, may be paid a commission of 25% of the selling price of the merchandise that he sells. If he sells $12,000 worth, his commission is 25% of $12,000.

If c represents the number of dollars earned, then the ratio of the number of dollars earned to the number of dollars of sales is $\frac{c}{12,000}$. The commission ratio is $\frac{25}{100}$.

$$\frac{c}{12,000} = \frac{25}{100}$$

$$100c = 25 \times 12,000$$

$$c = \frac{300,000}{100}$$

$$c = 3,000$$

The salesman earns $3,000 on his sales of $12,000.

The numbers in this problem are easy and you probably could think 25% is $\frac{1}{4}$ and $\frac{1}{4}$ of $12,000 is $3,000. Other situations will use numbers which do not lend themselves to short cuts. Hence, the above example was solved in the general way.

Sometimes the *percent* of the selling price which gives the commission is called the *rate* of the commission.

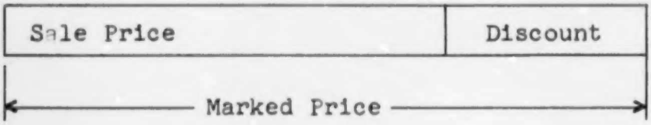
Definition: *Commission* is the payment, often based on a percent of the selling price, that is paid to a salesman for his services.

Merchants sometimes sell articles at a discount. During a sale, an advertisement stated "All coats will be sold at a discount of 30%." A coat marked $70.00 then has a discount of 30% of $70.00 or $21.00. The *sale price* (sometimes called the *net price*) is $70.00 - $21.00 or $49.00.

Definition: *Discount* is the amount subtracted from the marked price.

Definition: *Sale price* or *net price* is the marked price less the discount.

This relationship can be pictured as follows:



Sale Price + Discount = Marked Price

or

Marked Price - Discount = Sale Price

Example: Find the sale price of a coat marked \$80 if there is a discount of 30%. Choose d to represent the number of dollars in this discount.

$$\frac{d}{80} = \frac{30}{100}$$

$$100d = 2400$$

$$d = \frac{2400}{100}$$

$$d = 24$$

The discount is \$24.

The sale price is \$80 - \$24 or \$56.

Exercises 10-5b

In the following problems it may be necessary to round some answers. Round money answers to the nearest cent, and round percent answers to the nearest tenth of a percent.

1. On an examination there was a total of 40 problems. The teacher considered all of the problems of equal value, and assigned grades by percent. How many correct answers were indicated by the following grades?

   a. 100%   b. 80%   c. 50%   d. 65%

2. On the examination in Problem 1, what percent did the following students receive? Each grade is based on a total of 40 problems.

   a. Emma: All problems worked, but 10 answers wrong.
   b. Muriel: 36 problems worked, all answers correct.
   c. Don: 20 problems worked, 2 answers wrong.
   d. Bill: 1 problem not answered, and 1 answer wrong.

3. If the sales tax in a certain state is 4% of the purchase price, what tax is collected on the following purchases?

   a. A dress selling for \$17.50
   b. A bicycle selling for \$49.50

4. A real estate agent receives a commission of 5% for any sale that he makes. What is his commission on the sale of a house for $27,500?

5. The real estate agent in Problem 4 wishes to earn an annual income of at least $9,000 from his commissions. To earn this income, what must his yearly sales total?

6. A salesman who sells vacuum cleaners earns a commission of $25.50 on each one sold. If the selling price of the cleaner is $85.00 what is the rate of commission for the salesman?

7. Sometimes the rate of commission is very small. Salesmen for heavy machinery often may receive a commission of 1%. If, in one year, such a salesman sells a machine to an industrial plant for $658,000 and another machine for $482,000, has he earned a good income for the year? What is the income?

8. A sports store advertised a sale of football equipment. The sale discount was 27%.
    a. What was the sale price of a football marked at $5.98?
    b. What was the sale price of a helmet if the marked price was $3.75?

9. In Lincoln High School there are 380 seventh grade pupils, 385 eighth grade pupils, and 352 ninth grade pupils.
    a. What is the total enrollment of the school?
    b. What percent of the enrollment is in the seventh grade?
    c. What percent of the enrollment is in the eighth grade?
    d. What percent of the enrollment is in the ninth grade?
    e. What is the sum of the answers to b, c, and d?

10. Mr. Martin kept a record of the amounts of money his family paid in sales tax. At the end of one year he found that the total was $96.00 for the year. If the sales tax rate was 4%, what was the total amount of taxable purchases made by the Martin family during the year?

Percents Used for Comparison

In some of the problems we have met, the percents were not whole percents. The fraction $\frac{1}{8}$ changed to percent is $12\frac{1}{2}\%$. Also, you have used $\frac{1}{2}\%$ and have learned that $\frac{1}{2}\%$ may be read $\frac{1}{2}$ of 1%. Decimal percents such as 0.7% are also used.

$$0.7\% = 0.7 \times \frac{1}{100} = \frac{0.7}{100} = \frac{7}{1000} = 0.007$$

These are all names for the same number. If we wish to find 0.7% of \$300, we need to find n so that

$$\frac{n}{300} = \frac{0.7}{100}$$

$$100n = 210$$

$$n = 2.10$$

0.7% of \$300 is \$2.10

(0.7% is less than 1%. Since 1% of \$300 is \$3.00, the answer \$2.10 is sensible.)

Suppose that we wish to find 2.3% of \$500.

$$2.3\% = \frac{2.3}{100}$$

Find the number x such that

$$\frac{x}{500} = \frac{2.3}{100}$$

$$100x = 1150$$

$$x = 11.50$$

2.3% of \$500 is \$11.50

The game of baseball uses percent for making comparisons. A baseball batting "average" for a player is the ratio of the number of hits the player made to his number of times at bat. This ratio is expressed as a decimal and rounded to the nearest thousandth. The batting average can be considered as a percent expressed to the nearest tenth of a percent. If the player has

23 hits out of 71 times at bat, his batting average is $\frac{23}{71}$ or .324. This is 32.4%. In newspaper reports the decimal point in .324 is often omitted in order to save space.

Sometimes, grades are called for to the nearest tenth of a percent. A teacher may be asked what percent of the number of marks in his class are B's. Suppose he issued 163 marks, 35 of them B's. He wishes to find x such that

$$\frac{x}{100} = \frac{35}{163}$$

$$163x = 3500$$

$$x = \frac{3500}{163}$$

$$x = 21.47 \ldots$$

In the chapter on decimal notation you learned how to round decimals. If the percent is called for to the nearest tenth of a percent, 21.47 ... is rounded to 21.5%.

Exercises 10-5c

1. Find the batting averages of these four players. Who has the best average? Give answers as 3 place decimals and as percents rounded to the nearest tenth of a percent.

Player	Times at bat	Hits
George	65	19
Max	70	22
Bill	73	21
Tom	60	21

2. A nut grower has found that 4.6% of the nuts that he grows can be expected to be below standard. At this rate, how many pounds of nuts will be below standard in a bag weighing 80 pounds? Express your answer to the nearest tenth of a pound.

3. When Mr. Davis built his new home, he found that the cost of the land was 18.5% of the cost of the house. If the land cost $3,330, what was the cost of the house?

4. The Lewis family has an income of $5,200 a year. If they save $450 a year, what percent of their income is this? Express your answer to the nearest tenth of a percent.

Percents of Increase and Decrease

Central City had a population of 32,000 (rounded to the nearest thousand) in 1950. The population increased to 40,000 by 1960. What was the percent of increase?

The "percent of increase" asks for the ratio of the amount of increase to the original amount. In this case the increase is 40,000 - 32,000 or 8,000. The original population, or the population in the beginning, was 32,000. The problem is to find g so that

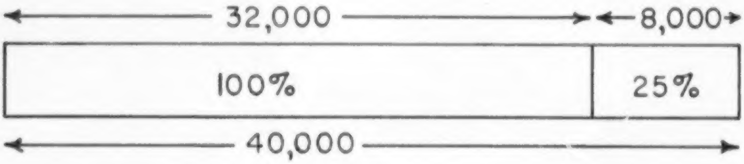
$$\frac{g}{100} = \frac{8{,}000}{32{,}000}$$

$$g = \frac{8{,}000 \times 100}{32{,}000}$$

$$g = 25$$

The percent of increase was 25%.

Notice that the percent of increase compares the actual increase to the original or earlier population number.



Hill City had a population of 15,000 in 1950. If the population in 1960 was 12,000, what was the percent of decrease? The actual decrease is 15,000 - 12,000 = 3,000. If x represents the percent of decrease, then

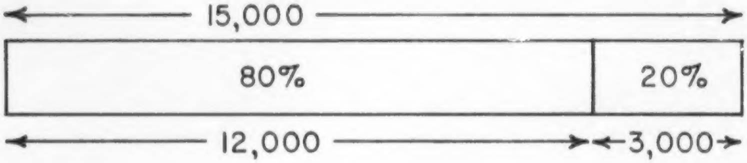
$$\frac{x}{100} = \frac{3,000}{15,000}$$

$$15,000x = 300,000$$

$$x = \frac{300,000}{15,000}$$

$$x = 20$$

The percent of decrease was 20%. Notice that the population decrease is computed by comparing the actual decrease with the original or earlier population figure.



If the rents in an apartment house are increased 5%, each tenant can compute his new rent. Suppose that a tenant is paying $80 for rent. What must he pay in rent after the increase? If x represents the increase in rent, then

$$\frac{x}{80} = \frac{5}{100}$$

$$100x = 400$$

$$x = 4$$

The increase is $4.00

The new rent will be $80 + $4 = $84.

Exercises 10-5d

1. There were 176 pupils in 5 mathematics classes. The semester marks of the pupils were A, 20; B, 37; C, 65; D, 40; E, 14. Give answers to the nearest tenth percent.
   a. What percent of the marks were A?
   b. What percent of the marks were B?
   c. What percent of the marks were C?
   d. What percent of the marks were D?
   e. What percent of the marks were E?
   f. What is the sum of the answers in parts a, b, c, d, e? Does this sum help you check your answers?
2. Bob's weight increased during the school year from 65 pounds to 78 pounds. What was the percent of increase?
3. During the same year, Bob's mother reduced her weight from 160 pounds to 144 pounds. What was the percent of decrease?
4. The enrollment in a junior high school was 1240 in 1959. In 1962 the enrollment had increased 25%. What was the enrollment in 1962?
5. Jean earned $14.00 during August. In September she earned only $9.50. What was the percent of decrease in her earnings?
6. A salesman of heavy machinery earned a commission of $4,850 on the sale of a machine for $970,000.
   a. Find his rate of commission.
   b. What can he expect as his commission for the sale of another machine for $847,500?
7. James was 5 ft. tall in September. In the following June his height was 5 ft. 5 in. Both heights were measured to the nearest inch. What was the percent of increase in his height?

8. Do you know what your height was at the beginning of the school year? Do you know what it is now? Do you know what your weight was at the beginning of the school year? Now?

   a. What is the percent of increase in your height since last September?

   b. What is the percent of increase in your weight since last September?

9. A baseball player made 25 hits out of 83 times at bat. Another player made 42 hits out of 143 times at bat.

   a. What is the batting average of each player?

   b. Which player has the better record?

10. An elementary school had an enrollment of 790 pupils in September, 1960. In September, 1961, the enrollment was 1012. What was the percent of increase in enrollment?

11. On the first day of school a junior high school had an enrollment of 1050 pupils. One month later the enrollment was 1200. What was the percent of increase to the nearest tenth of a percent?

12. One week the school lunchroom receipts were $450. The following week the amount was $425. What was the percent of decrease to the nearest tenth of a percent?

13. A baby's weight usually increases 100% in his first six months of life.

   a. What should a baby weigh at six months, if its weight at birth was 7 lb. 9 oz.?

   b. Suppose that the baby in Part (a) weighs 17 lbs. at the age of six months. What is the percent of increase to the nearest percent?

14. During 1960 a family spent $1,490 on food. In 1961 the same family spent $1,950 on food. What was the percent of increase in the money spent for food to the nearest percent?

*15. During 1958 the owner of a business found that sales were below normal. The owner announced to his employees that all wages for 1959 would be cut 20%. By the end of 1959 the owner noted that sales had returned to the 1957 levels. The owner then announced to the employees that the 1960 wages would be increased 20% over those of 1959.

Which of the following statements is true?

a. The 1960 wages are the same as the 1958 wages.

b. The 1960 wages are less than the 1958 wages.

c. The 1960 wages are more than the 1958 wages.

Interest

The charging and payment of interest are important in business affairs. You may have noticed this. Savings banks advertise that they pay 3% or 4% on savings. Lending companies advertise that they will lend money at 6% or 7%. The numbers change but the ideas of percent remain the same.

Everyone knows that if you live in a house or apartment which belongs to someone else you pay for this privilege. The fee you pay is called rent.

In the same way, if you arrange to use money which belongs to someone else, you pay for this privilege. The fee you pay is called interest. If you have money in a savings account you receive the interest, for then you are the lender. The rate of interest when expressed as a percent is usually considered the rate for one year.

The amount of money upon which the interest is paid is called the principal. The ratio of the interest to the principal is called the percent (or rate) of interest.

Example: At the end of the year Marie received $13.50 interest on her savings account of $450. Her bank paid 3% on savings for one year. $450 is the principal. $13.50 is the interest. 3% is the rate.

a. How do you find the rate when you know the interest and the principal? The rate is a percent. Let $\frac{r}{100}$ represent the ratio. This is the same as the ratio of interest to principal. So

$$\frac{r}{100} = \frac{13.50}{450}$$

$$r = \frac{13.50 \times 100}{450}$$

$$r = 3$$

The rate is 3%.

b. How do you find the interest when you know the rate and the principal?

The rate is 3% or $\frac{3}{100}$. The interest is n dollars and the ratio of the interest to the principal is $\frac{n}{450}$.

$$\frac{3}{100} = \frac{n}{450}$$

$$\frac{3 \times 450}{100} = n$$

$$13.50 = n$$

The interest is $13.50

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c. How do you find the principal when you know the rate and the interest?

The rate is 3% or $\frac{3}{100}$. The interest is \$13.50 and the ratio of the interest to the principal, p dollars, is $\frac{13.50}{p}$.

$$\frac{3}{100} = \frac{13.50}{p}$$

$$3p = 1350$$

$$p = 450$$

The principal is \$450.

Exercises 10-5e

1. Find the interest for one year on \$1,800 at 4%.
2. What is the rate if the interest for one year on \$1,250 is \$75?
3. What amount of interest will \$900 earn in a year at $3\frac{1}{2}$% ?
4. What principal will earn \$42.50 in a year at 5%?
5. Find the interest on \$250 at 4% for a year.
6. Find the interest earned in one year on \$3,500 at 3%.
7. The interest for one year on \$800 is \$28. What is the rate?
8. Mildred earned \$1.80 in one year on a sum of money she deposited at $4\frac{1}{2}$%. How much did she deposit? (That is, find the principal.)
9. Veryl deposited \$350 in a bank which paid 4% interest per year. At the end of the year she decided to leave her interest in the bank. How much did she then have, altogether, in her account?

*10. In Problem 9, Veryl kept her money in the same bank for another year. How much did she then have at the end of this second year?

____

10-6. Summary.

In order to prepare for the ideas of percent, you learned about ratio and proportion. You will use the principles of ratio and proportion in your later work in geometry and in numerous applications to science problems as well as to problems of business.

Percent is widely used in practical affairs. Everyone must know how to deal with problems involving percent. Only selected examples of the use of percent can be included in this chapter, but any percent problem can be solved by the methods developed in this chapter. All you need to know is how to translate the facts of the problem into a proportion and then solve the proportion.

The ratio of a number c to a number d $(d \neq 0)$ is the quotient $\frac{c}{d}$.

The Comparison Property: If $\frac{a}{b} = \frac{c}{d}$ $(b \neq 0,\ d \neq 0)$ then $a \cdot d = b \cdot c$.

Any fraction, $\frac{a}{b}$, can be expressed as a percent by finding c so that $\frac{a}{b} = \frac{c}{100} = c\%$.

Any fraction, $\frac{a}{b}$, can be written in decimal form and then expressed as a percent by finding c so that $\frac{a}{b} = c \times .01 = c\ \%$.

The sign $\approx$ means approximately equal.

A budget is a plan for spending money. The ratios of the separate parts to the total income are often expressed as percents.

A commission is the payment to a salesman for his service. It is often based on a percent of the amount of his sales.

A discount is the amount subtracted from the marked price.

The sale price is the marked price less the discount.

The percent of increase or decrease is the ratio of the amount of increase or decrease to the original amount.

Interest is a fee paid for the use of money.

The principal is the amount of money on which interest is paid. It is the amount borrowed or loaned.

The rate of interest is the ratio of the interest to the principal. The rate of interest is stated as a percent.

10-7. Chapter Review.

Exercises 10-7

1. Write the ratio of

   a. 22 to 24

   b. 63 to 56

2. Which of the following ratios are equal?

   a. $\frac{22}{11}, \frac{8}{4}$

   b. $\frac{14}{7}, \frac{18}{8}$

   c. $\frac{48}{16}, \frac{54}{18}$

   d. $\frac{18}{17}, \frac{72}{68}$

3. In each of the following find the number for c which will make the statement true.

   a. $\frac{20}{6} = \frac{c}{9}$ c. $\frac{3}{c} = \frac{15}{75}$

   b. $\frac{c}{90} = \frac{19}{30}$ d. $\frac{81}{100} = \frac{12}{c}$

4. A picture 4 inches wide and $4\frac{1}{2}$ inches high must be reduced to fit a 2 inch newspaper column. How high will the picture in the newspaper be?

5. A triangle has sides of length 11 inches, 8 inches, and 6 inches. In another triangle, the measures of the sides have the same ratio. The shortest side of the second triangle is 9 inches in length. Find the length of each of the other sides.

6. Write the following as percents.

   a. .02   b. .055   c. $\frac{2}{3}$ d. $\frac{1}{8}$

7. The monthly take-home pay for the Donovan family is $400. The payment on the mortgage for their house is $80 a month. What percent of the Donovan income is needed to pay the mortgage?

8. If the 4% sales tax on a new car was $108, what was the price of the car, not including the tax?

9. The records of Central High School show that about 76% of the students can be expected to buy the yearbook. The enrollment this year is 1862. How many pupils can be expected to buy the book this year?

10. Mr. Stephens earns a commission of 40% of his sales. His sales in October amounted to $2,450. What was his commission in October?

11. Boys' sweaters marked at $12.00 were on sale at a discount of 30%. Find the sale price.

12. When calls on coin telephones increased in price from 5 cents to 10 cents, what was the percent of increase?

13. State the property of equal rational numbers which is useful in solving a proportion.

    Use $\frac{15}{27} = \frac{20}{36}$ in your statement.

14. The baseball team of Room 106 won 13 games of the 16 the boys played. The Room 107 team won 15 games of the 18 they played. Which team had the better record?

15. Find the interest on $1,050 at 3% for one year.

10-8. Cumulative Review.

Exercises 10-8

1. True or False:
   a. Every number can be completely factored in only one way.
   b. In the base twelve system $7 \times 9 = 53_{\text{twelve}}$.
   c. The numeral "4" (four) has the same meaning in the base five system of numeration as in the base ten system of numeration.

2. In which base has this multiplication been performed?

$$\begin{array}{r} 123 \\ 32 \\ \hline 312 \\ 1101 \\ \hline 11322 \end{array}$$

3. Write the numeral for 17 in
   a. Base five
   b. Base eight
   c. Base two

4. Use the associative property of multiplication so that this product can be found easily:

$$31 \cdot 5 \cdot 2$$

5. Write the following statements in words:

   a. $8 < 12$

   b. $34 > 32$

   c. $5 > 3 > 2$

6. State the commutative property for addition using the symbols a and b for any whole numbers.

7. The following points are suggested by objects in your kitchen. Point A is suggested by the refrigerator handle. Point B is suggested by a foot of the kitchen table. Point C is suggested by the faucet on your kitchen sink.

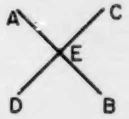
   a. How many different planes contain A and B?

   b. How many different lines contain A and C?

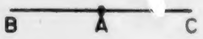
   c. How many different planes contain A, B, and C?

8. In each case, describe the union of the two sets given below:

   a. The set of points on the C side of $\overleftrightarrow{AB}$ and the A side of $\overleftrightarrow{CD}$.



   b. The set of points on $\overrightarrow{AB}$ and the set of points on $\overrightarrow{AC}$.



9. Write the larger of each of the following pairs.

   a. $\frac{2}{6}$, $\frac{4}{10}$

   b. 13, $\frac{67}{8}$

   c. $\frac{9}{11}$, $\frac{11}{17}$

10. Perform the following operations.

   a. $(6.04 + 5.073) - 2.909$

   b. $(8.326 - 3.041) - 2.998$

11. Write the following as percents.

a. $\frac{4}{5}$

b. .026

c. $\frac{5}{6}$

d. $\frac{3}{8}$

12. The Fairlawn basketball team won 18 games of the first 25 played. The team won 5 of the next 9 games. Did the team improve its record?

13. Perform the indicated operations and simplify:

a. $\frac{4}{7} \times 7$

b. $4 \times \frac{11}{4}$

c. $\frac{1}{2} \div \frac{1}{2}$

d. $\frac{0}{3} \times \frac{12}{5}$

e. $\frac{13}{7} \div \frac{11}{11}$

f. $\frac{1}{3} \div 2$

g. $6 \div \frac{1}{2}$

h. $\frac{5}{6} \div 5$

i. $10 \div \frac{1}{6}$

j. $30 \div \frac{3}{10}$

14. Find the following products.

a. $.25 \times .03$

b. $.002 \times .7$

c. $1.2 \times .35$

d. $3.26 \times .04$

15. Find the following quotients.

a. $.0312 \div .3$

b. $2.35 \div 5$

c. $12.08 \div .08$

d. $.612 \div .4$

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